Cosmic Strings and Filaments:
Unraveling the Subtleties of Femtolensing and Accretion in Cylindrically Symmetric Spacetime

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Executive Summary

Einstein’s theory of general relativity, one of the great accomplishments of physics in the last century, has been used to study a huge range of astrophysical phenomena since its birth in 1915. General relativity models the effects of gravity and motion in gravitational fields as motion on a curved manifold called spacetime, the curvature of which is coupled to the mass and energy present in that spacetime via a set of nonlinear, coupled partial differential equations known as Einstein’s equations.

These equations are very difficult to solve. Despite this, many exact solutions have been found, most of which employ some form of symmetry to simplify the problem. A vast portion of the phenomena studied using relativity have been ones with spherical symmetry. This is primarily because many astrophysical objects have some form of spherical symmetry. This is not to say that all astrophysical phenomena have this type of symmetry. For instance, many theories have predicted the existence of cosmic strings, topological defects in spacetime that can be modeled with cylindrical symmetry.

Tullio Levi Civita found a general solution for a stationary, cylindrical spacetime, now known as the Levi-Civita spacetime. In this thesis, we will discuss some of the subtleties
in dealing with cylindrically symmetric spacetime, and apply techniques of gravitational lensing in order to prove that the linear mass density of a cosmic string or other long, thin filament of matter could in principle be determined using a technique known as femtolensing. Femtolensing occurs when the effects of gravitational lensing are too weak to produce any observable distortion in the amplitude of light detected from a source, but the phase is still altered in a characteristic way. In the context of femtolensing from a filament, we expect the two beams of light, traveling above and below the filament, to interfere with each other in a manner characteristic of the linear mass density of the filament. We will also show how the standard techniques of relativistic hydrodynamics can be used to determine the accretion of dust around such a filament.
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Chapter 1

Introduction and a Brief History

Albert Einstein’s theory of General Relativity, put forth in 1915, was an astronomical leap forward in our understanding of the universe. It succeeded Newtonian gravity, as well as revolutionizing our understanding of space, time, and energy. The notion of relativity in physics is, in fact, even older than Newton’s theory of gravity. In 1632 [2], Galileo Galilei suggested that an observer moving at a constant velocity, such as one aboard a smooth sailing vessel, must experience the same laws of physics as one on does on land. This notion that all inertial observers must experience indistinguishable laws of physics, known as Galilean Invariance, was one of the key precursors to Einstein’s theory of relativity.

The development of a unified theory of electricity and magnetism, culminating in Maxwell’s electromagnetism in the nineteenth century, provided a more complete theory of light. Maxwell’s electromagnetism required that light propagate at a constant speed, which in the context of relativity, implied the presence of some exterior reference frame in which
the speed of light is defined absolutely. At the time, it was believed this reference frame was defined by some invisible and intangible medium in which light propagates, known as the luminiferous aether. Even after the presence of this aether was disproven by Michelson and Morley in their famous 1887 experiment, no explanation for the behavior of light would be accepted until Einstein published his theory of special relativity in 1905.

Albert Einstein, in a paper titled On the Electrodynamics of Moving Bodies [3], proposed a theory in which space and time should not be regarded as absolute, but as different manifestations of a greater “spacetime”. This theory was derived from the two simple postulates:

- The laws of physics are identical in all inertial frames
- The speed of light (in a vacuum) is the same for all observers, in all reference frames.

The former of these two postulates is simply Galileo’s principle of relativity. The second was by all means the more radical suggestion. With the resulting theory, Einstein provided a long awaited solution to the problem of the luminiferous aether: that it is not needed at all, provided one is willing to throw away a notion of absolute space and absolute time that Newtonian physics holds dear.

But with the discarding of Newtonian space and time, and special relativity only describing inertial frames, a race began to develop a theory that would describe the physics on non-inertial (that is, accelerated) frames, such as those under the effects of gravity. Many physicists, such as Poincaré, Nordström, Abraham, and Mie developed scalar theories to describe the effects of gravitation in a way consistent with special relativity. All of these theories of gravity failed in some way, until Einstein published again in 1915 with a tensor
theory, describing spacetime as the curvature of a four dimensional manifold, spacetime. Not only did his theory reduce to special relativity, but it also made predictions about astrophysical effects, such as the deflection of light as it travels near the sun, which was confirmed to great accuracy in the following years. This is the theory now known as general relativity, for which the theory of special relativity is a “special” case.

As will be discussed in the next chapter, the equations of general relativity are in no way easy to solve. Einstein’s field equations, which relate the curvature of spacetime to the distribution of matter and energy, are a set of ten highly non-linear coupled differential equations, and these equations have been studied extensively but not exhaustively in the 101 years since its publication.
CHAPTER 1. INTRODUCTION AND A BRIEF HISTORY
Chapter 2

Curved Spacetime and General Relativity

2.1 A Brief Note on Notation

The field of general relativity is notoriously plagued by inconsistent notation. Because of this, I will presently declare the notations I have chosen.

- Vector indexed objects require specification of both components and basis vectors, but for compactness vectors will be expressed with a superscript signifying the component, neglecting the basis vector. For example, the vector \( \vec{V} \) is specified as

\[
\vec{V} = \sum_i V^i \vec{e}_i
\] (2.1)
• Greek vs. Latin Indices: Because we will be juggling four-vectors and three-vectors quite a bit, it is useful to put restrictions on our indexing. From here on out, Latin letter indicies (most frequently $i$ and $j$) will be assumed to range over the spatial indicies (1, 2, and 3), while Greek letter indicies (most frequently $\mu$, $\nu$, $\rho$, $\sigma$, $\lambda$, and $\gamma$) will be assumed to range over all spacetime indicies (0, 1, 2, and 3).

• Einstein Summation Notation: Because of the extensive use of summation in relativity, it is standard to express the sum over indicies as

$$\sum_i V_i V^i = V_i V^i$$

meaning that a sum over the full range of the $i$ index is implied, even if a summation symbol is left out, whenever the same index letter appears twice in a single algebraic term.

• Units: We will be working in units where $c = G = 1$, so don’t worry if the speed of light doesn’t appear much. It is still very much there, making everything either really big or really small relative to SI units. Factors of $G$ and $c$ can be reinserted easily by dimensional analysis.

2.2 Geodesics and Inertial Frames

General relativity is a tensor field theory, describing gravitation not as a conventional force but as free motion on a pseudo-Riemannian manifold, the curvature of which is coupled to mass and energy via Einstein’s equations. Mathematically, a manifold refers to a set of points, each of which can be labeled with coordinates. In many cases, and particularly in
2.2. GEODESICS AND INERTIAL FRAMES

In general relativity, these coordinate systems are taken to be fundamentally arbitrary, but they can often be adapted to the symmetries of a particular problem, as we will see in this thesis. A manifold is considered Riemannian or pseudo-Riemannian if there is a particular kind of rule for defining the length of an arbitrary curve. This rule is often called the metric structure, though in formulas it’s usually clearest to write it in terms of an object called the line element, which defines the length of an infinitesimal curve, which can be extended to a curve of arbitrary length via integration. For example, the line element of two-dimensional Euclidean space is:

\[ ds^2 = dx^2 + dy^2 \]  

(2.3)

If a curve is defined parametrically through functions \( x(\lambda), y(\lambda) \), where \( \lambda \) is an arbitrary parameter, then the line element implies:

\[ \left( \frac{ds}{d\lambda} \right)^2 = \left( \frac{dx}{d\lambda} \right)^2 + \left( \frac{dy}{d\lambda} \right)^2 \]  

(2.4)

\[ \Delta s = \int \sqrt{\dot{x}^2 + \dot{y}^2} d\lambda \]  

(2.5)

where overdots refer to derivatives with respect to \( \lambda \).

The presence of a functional definition of arclength provides a foundation for describing the geometry of the manifold. Specifically, whereas in Euclidean space the shortest path between two points is a straight line, in a general Riemannian manifold, a straight line, more generally known as a geodesic, is defined as the path between two points with (locally) minimum length. A pseudo-Riemannian manifold is one where the line element is not positive-definite. If the separation \( ds^2 \) between two events on such a manifold is negative, we say the events are timelike separated. If \( ds^2 > 0 \), we say the events are spacelike.
CHAPTER 2. CURVED SPACETIME AND GENERAL RELATIVITY

separated. Finally, if \( ds^2 = 0 \), the events are null separated. Curves on the manifold that connect timelike separated events are known as timelike curves. Spacelike curves and null curves are defined similarly.

In an arbitrary manifold, in arbitrary coordinates, the line element is more generally described by the equation

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu.
\]  
(2.6)

where \( g_{\mu\nu} \) is a matrix valued object known as the metric tensor. This relationship comes simply from the metric’s definition. The metric tensor is defined by the scalar product of two vectors, generally written as

\[
\vec{U} \cdot \vec{V} = g_{\mu\nu} U^\mu V^\nu.
\]  
(2.7)

So the line element \( ds^2 \) can be written:

\[
ds^2 = d\vec{x} \cdot d\vec{x} = g_{\mu\nu} dx^\mu dx^\nu
\]  
(2.8)

as desired.

We can now use this notion of path length to develop a mathematical definition of a geodesic, a ‘straight line in a curved geometry.’ Consider, as in equation 2.5, that our path is parametrized by some parameter \( \lambda \),

\[
\Delta s = \int \sqrt{|g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu|} d\lambda
\]  
(2.9)

where once again the overdot represents differentiation with respect to \( \lambda \). If we extremize
2.2. GEODESICS AND INERTIAL FRAMES

this as an action, we arrive at Euler-Largange equations:

\[
\frac{\partial}{\partial \lambda} \frac{\partial}{\partial \dot{x}^\gamma} \sqrt{|g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu|} - \frac{\partial}{\partial x^\gamma} \sqrt{|g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu|} = 0.
\] (2.10)

In the case where \( \lambda \) is an affine parameter, this can be shown [4] to reduce to

\[
0 = \ddot{x}^\rho + \frac{1}{2} g^{\rho \sigma} (\partial_\mu g_{\nu \sigma} + \partial_\nu g_{\sigma \mu} - \partial_\sigma g_{\mu \nu}) \dot{x}^\mu \dot{x}^\nu
\] (2.11)

\[
0 = \ddot{x}^\rho + \Gamma^\rho_{\mu \nu} \dot{x}^\mu \dot{x}^\nu
\] (2.12)

where \( \partial_\mu = \frac{\partial}{\partial x^\mu} \). The three indexed factor attached to the second term has a name: the Christoffel symbol, defined as

\[
\Gamma^\rho_{\mu \nu} = \frac{1}{2} g^{\rho \sigma} (\partial_\mu g_{\nu \sigma} + \partial_\nu g_{\sigma \mu} - \partial_\sigma g_{\mu \nu}).
\] (2.13)

Equations 2.11 and 2.12 are two forms of the geodesic equation. We can see that if \( \Gamma^\rho_{\mu \nu} \) are all equal to zero, then we are left with \( \ddot{x} = 0 \), Newton’s second law for a free particle.

Frames where this property is satisfied are known as inertial frames.

Flat spacetime is described by the Minkowski metric, defined as

\[
g_{\mu \nu} = \eta_{\mu \nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\] (2.14)
so the line element becomes

\[ ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \]  \hspace{1cm} (2.15)

For this metric, equation 2.13 immediately implies that \( \Gamma^\rho_{\mu\nu} \) vanish. It is from this metric that the laws of special relativity can be derived. A few more tools must be developed in order to describe physics in spacetimes with curvature.

Alongside this understanding of spacetime as a pseudo-Riemannian manifold comes a new manifestation of the principle of equivalence. Einstein’s self-proclaimed happiest thought was that in a freely falling reference frame, the laws of physics must be indistinguishable from those of special relativity. In the language of general relativity, this is to say that at any event \( P \), there must be a set of coordinates of the form \( \xi^\mu \) where the spacetime locally resembles the Minkowski line element,

\[ ds^2 = \eta_{\mu\nu} d\xi^\mu d\xi^\nu. \]  \hspace{1cm} (2.16)

It can be shown \[4\] that such a coordinate system can always be defined within a neighborhood of any event in spacetime. Such “Riemann Normal Coordinates” provide the mathematical underpinning of Einstein’s equivalence principle.

### 2.3 Constructing the Riemann Curvature Tensor

Each point on a manifold has its own tangent space, in which vectors are defined. One unfortunate feature of describing spacetime as a manifold is that the partial derivative is
not a good tensor operator, that is to say taking the partial derivative of a tensor field does not preserve its tensor properties in translating between these tangent spaces.

A clearer way to motivate the need for a new derivative operator is to look once more at vectors. Consider taking the partial derivative of a vector field $\vec{V}$. What we really mean when we say $\partial_\mu \vec{V}$ is

$$\partial_\mu \vec{V} = \partial_\mu (V^\alpha \vec{e}_\alpha)$$

(2.17)

where $V^\alpha$ are the vector components and $\vec{e}_\alpha$ are the basis vectors defined in the tangent space. The product rule can now be applied to show that

$$\partial_\mu \vec{V} = \partial_\mu (V^\alpha \vec{e}_\alpha) = \vec{e}_\alpha \partial_\mu V^\alpha + V^\alpha \partial_\mu \vec{e}_\alpha$$

(2.18)

However, the derivatives of the basis vectors constitutes another vector field that can be expressed in terms of the original basis, leaving

$$\partial_\mu \vec{V} = \vec{e}_\alpha \partial_\mu V^\alpha + \Gamma^\alpha_{\mu\nu} V^\nu \vec{e}_\alpha$$

(2.19)

where $\Gamma^\rho_{\mu\nu}$, the bookkeeping coefficients relating the derivatives of the basis vectors back to the original basis, are the very same Christoffel symbols we derived when talking about geodesics in the previous section! It therefore makes sense to define a modified differential operator, called the ‘covariant derivative’ operator, that takes these extra terms into account when working with vector components. The operation of the covariant derivative $\nabla_\mu$ operating on some set of vector components $V^\nu$ is defined as

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$$

(2.20)
An important feature of this derivative is that it does not necessarily commute. In fact, in curved spacetime the second covariant derivative of a vector fails to commute in a very interesting manner:

\[ \nabla_\mu \nabla_\nu V^\alpha - \nabla_\nu \nabla_\mu V^\alpha = \frac{1}{2} R^\alpha_{\beta \mu \nu} V^\beta \]  

(2.21)

where \( R^3_{\beta \mu \nu} \) is a rank 4 tensor defined as

\[ R^\mu_{\nu \alpha \beta} = \partial_\alpha \Gamma^\mu_{\nu \beta} - \partial_\beta \Gamma^\mu_{\nu \alpha} + \Gamma^\mu_{\lambda \alpha} \Gamma^\lambda_{\nu \beta} - \Gamma^\mu_{\lambda \beta} \Gamma^\lambda_{\nu \alpha}. \]  

(2.22)

This is the Riemann Curvature Tensor. It allows us to define the Ricci tensor, a rank two symmetric tensor generated by contracting the Riemann Curvature tensor,

\[ R_{\mu \nu} = R^\lambda_{\mu \lambda \nu} \]  

(2.23)

We can also construct a scalar by contracting over the remaining indices. The scalar generated by doing so is known as the Ricci Scalar, or the Curvature Scalar:

\[ R = R^\mu_{\mu} = g_{\mu \alpha} R^\alpha_{\mu \nu}. \]  

(2.24)

Now, at long last with this expansive arsenal of tools, we can write down Einstein’s field equations. These are the equations that govern the influence that matter and energy have on the curvature of spacetime. The equations are commonly written in the form

\[ G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = 8 \pi G T_{\mu \nu} \]  

(2.25)
2.3. CONSTRUCTING THE RIEMANN CURVATURE TENSOR

Where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is called the Einstein tensor and $T$ is known as the stress-energy tensor. These equations, if written out in full using all the definitions above, constitute a coupling of the metric coefficients to the matter and energy presence described by the components of stress-energy via highly non-linear partial differential equations.

For reference, the stress energy tensor can take many forms, but there are a few features worth noting. First, the $T^{00}$ component often represents the density of relativistic mass. The components $T^{0i} = T^{i0}$ represent a flux of relativistic mass and $T^{ik} = T^{ki}$ represent material stresses, such as the pressure of a fluid, or Maxwell’s stress tensor of an electromagnetic field.

Once again, this glance at general relativity has been brief, and is mainly meant to fix notation and to summarize intuitive background. For a more thorough treatment, a wealth of textbooks exist [4] [5].
Chapter 3

Cylindrical Solutions

Einstein’s equations are a series of ten nonlinear coupled partial differential equations, and finding exact solutions is a notoriously difficult task. Thankfully, physicists have been discovering metric solutions with a variety of different properties for the past hundred years. Their tasks are typically simplified dramatically by applying coordinate symmetries to the metric. Solutions with spherical symmetry are perhaps the most thoroughly analyzed for their astrophysical utility. One of the most important solutions is the Schwarzschild solution:

\[
    ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2
\]  

(3.1)

This metric has been studied a great deal, and will not be discussed much here.
3.1 The Weyl Metric

Another particularly well studied class of spacetimes is that of axially symmetric metrics. Hermann Weyl and Tullio Levi-Civita in particular worked to find static solutions (that is, spacetimes that are irrotational and do not depend on time) with axial symmetry. In 1917 [6], Hermann Weyl showed that the general static, axisymmetric line element can be written as

\[ ds^2 = -e^{2U}dt^2 + e^{2\gamma-2U}d\eta^2 + e^{2\gamma-2U}d\xi^2 + \rho^2 e^{-2U}d\phi^2 \]  

(3.2)

where the potentials \( U, \gamma, \) and \( \rho \) depend only on \( \eta \) and \( \xi \). Weyl noted the convenient choice of coordinates \( \rho \) and \( z \) in which the above metric can be written as

\[ ds^2 = -e^{2U}dt^2 + e^{2\gamma-2U}d\rho^2 + e^{2\gamma-2U}dz^2 + \rho^2 e^{-2U}d\phi^2 \]  

(3.3)

where \( U \) and \( \gamma \) now depend solely on \( \rho \) and \( z \). In such spacetimes, \((\rho, \phi, z)\) can be interpreted as cylindrical coordinates.

In order to determine some sort of intuition for what this metric, and similar metrics which we will soon look at, might represent astrophysically, consider the limit as \( U \) becomes small. The time-time component of the metric can be written as

\[ g_{tt} = -e^{2U} = -\sum_{n=0}^{\infty} \frac{(2U)^n}{n!} \approx -(1 + 2U) \]  

(3.4)

This is the signature of Newtonian potential in linearized gravity, so we can interpret \( U \)

---

1In technical terms, this is a representation of the general class of metrics with commuting, hypersurface-orthogonal isometry generators, denoted by the coordinate vector fields \( \delta_t \) and \( \delta_\phi \).
as analogous to the Newtonian potential $\Phi_G$. This is because, in the weak field limit, this potential $U$ results in the geodesic equation taking the form (for a test mass):

$$\ddot{a} = \nabla U.$$  

(3.5)

Equipped with this intuition, we can take some well understood Newtonian potential $\Phi_G$ and plug it into equation 3.3 and attempt to interpret it as the fully relativistic analog to the linearized case with that potential. I say “attempt” because the Weyl solutions generated by plugging in Newtonian potentials as $U$ often come with a slew of subtleties that must be dealt with to preserve an intuition for the spacetime’s astrophysical meaning.

### 3.2 The Levi-Civita Metric

If we are considering a spacetime that might be akin to an infinite line mass, a good place to start is with the Newtonian potential for an infinite line mass, $\Phi = 2\lambda \log(\rho)$, where $\lambda$ is the mass per unit length. We also expect this metric to have an additional symmetry in the $z$-direction, meaning that $U$ and $\gamma$ are independent also of $z$. Tulio Levi-Civita, in 1916 [6] considered the case in which the potential could be written as $U = 2\lambda \log(\rho)$, and, where $\lambda$ is a constant. Then it can be shown that Einstein’s equations require $\gamma = 4\lambda^2 \log(\rho) + \log(k)$.

While this metric was written down in 1916 by Levi-Civita, its physical interpretation is plagued with many subtleties and was not well understood until the 1960s [6]. Some rewriting and rescaling is in order to clarify how this spacetime should be interpreted. Given
these potentials, the metric takes the form of

\[
ds^2 = -\rho^{4\lambda} dt^2 + k^2 \rho^{4\lambda(2\lambda-1)} d\rho^2 + \rho^{4\lambda(2\lambda-1)} dz^2 + \rho^{2(1-2\lambda)} d\phi^2. \tag{3.6}
\]

If we are to consider the coordinate \(\phi\) to be periodic, it is necessary to introduce an additional parameter \(D\) such that the metric takes the form

\[
ds^2 = -\rho^{4\lambda} dt^2 + \rho^{4\lambda(2\lambda-1)} d\rho^2 + \rho^{4\lambda(2\lambda-1)} dz^2 + D^2 \rho^{2(1-2\lambda)} d\phi^2 \tag{3.7}
\]

where the \(\phi\) coordinate is now defined to range from zero to \(2\pi\). In fact, Levi-Civita showed that this pair of potentials results in the most general form of a cylindrical, static spacetime [6]. The parameters \(\lambda\) and \(D\) must be considered carefully before attributing them any physical significance in this framework. However, even before we consider those parameters, there is a more striking issue with the above metric: it doesn’t make dimensional sense! We expect \(ds\) to have units of length, requiring that \(\rho\) is dimensionless. But we also hope to interpret \(\rho\) as the coordinate analog to \(\rho\) that appears in cylindrical coordinates as the distance away from the axis. To resolve this, we can rescale the variables with some arbitrary distance \(\rho_0\) such that \(3.7\) takes the form

\[
ds^2 = - \left( \frac{\rho}{\rho_0} \right)^{4\lambda} dt^2 + \left( \frac{\rho}{\rho_0} \right)^{4\lambda(2\lambda-1)} d\rho^2 + \left( \frac{\rho}{\rho_0} \right)^{4\lambda(2\lambda-1)} dz^2 + D^2 \left( \frac{\rho}{\rho_0} \right)^{2(1-2\lambda)} \rho_0^2 d\phi^2 \tag{3.8}
\]
3.3 The Deficit Angle and Cosmic Strings

Before we can attribute intuitive meaning to the metric as a whole, we must consider the physical meaning of this parameter $D$, since we have introduced it purely as a mathematical tool to enforce a periodic condition on one of our coordinates. To deduce its meaning, consider the case of $\lambda = 0$. Since we suspect the quantity $\lambda$ represents an analog to linear mass density, this condition should represent the absence of a massive filament. The metric becomes

$$ds^2 = -dt^2 + d\rho^2 + dz^2 + D^2 \rho^2 d\phi^2$$  \hspace{1cm} (3.9)

In this spacetime, consider a path of constant $\rho$, $z$, and $t$. We expect this to result in a loop of circumference $\int ds = 2\pi \rho$. If we carry out this integration process, we arrive at

$$\int ds = \int_0^{2\pi} \rho D d\phi = 2\pi D \rho.$$  \hspace{1cm} (3.10)

So the quantity $D$ acts as some kind of quantity defining a deficit angle, that changes the rotational geometry of the spacetime from familiar Euclidian geometry (where $D = 1$) to a more general form (where $D \neq 0$). Geometry in the $t = 0$, $z = 0$ plane is like geometry on a cone, in which circumference is less than the usual Euclidean value $2\pi \rho$, by a factor $D$.

While there has not been any observational confirmation of spacetimes of this form existing in nature, objects known as “Cosmic Strings” have been predicted to exist by theories of early universe cosmology and super-string theory [1].
Chapter 4

Gravitational Lensing and Femtolensing

One prediction of Einstein’s general theory of relativity which has proven to be of great astrophysical importance, both historically and recently, is that light rays should be deflected as they move through curved spacetime. This effect is known as gravitational lensing. It provided early observational evidence for general relativity, particularly a famous expedition led by Arthur Eddington in 1919, observing aberration in star positions near the sun during a total solar eclipse \cite{7}. It has been studied in many contexts since then.

Gravitational lensing is typically subdivided into three categories: strong lensing, weak lensing, and microlensing. Strong lensing occurs when the field surrounding the object generates multiple images of a light source. One phenomenon associated with strong lensing are Einstein Rings, which occur when a light source is distorted into a ring-like shape by an
object with rough spherical symmetry. An image of this type of phenomenon is presented in figure 4.1. Weak lensing is characterized by smaller distortions, where the lensing effect is not strong enough to create multiple images. These smaller distortions can take the form of stretching or magnification of objects behind the lens. Finally, there is microlensing, which occurs when the distortions are so weak that no distinguishable change in the image shape is detectable, but the lensing causes the image to brighten due to additional light bent towards the detector.

Each of these lensing categories has been studied in great detail, leading to advances in our understanding of cosmic structure, dark matter, and much more. More recently, a fourth category of lensing has emerged, known as femtolensing. A lensing phenomenon can be characterized as femtolensing when the angular deflection cannot be resolved by the detector, and where the brightening due to microlensing is insignificant or otherwise impractical. In this case, when the amplitude of the light bears little fruit, we turn to the phase. If the light emitted is coherent, multiple rays of light that have traveled different pathlengths can arrive at the same detector, recorded on the same pixel, and they can interfere based on their difference in phase. Femtolensing has been applied to the search for low-mass primordial black holes [8] and even the search for massless (i.e. $\lambda = 0, D \neq 0$) cosmic strings [1]. This is the effect that we will consider in the following sections.

### 4.1 Femtolensing in the Levi-Civita Spacetime

The primary topic I wish to investigate in this thesis is the possibility of femtolensing in a cylindrically symmetric spacetime such as the Levi Civita spacetime (equation 3.7) as
4.1. FEMTOLENSING IN THE LEVI-CIVITA SPACETIME

Figure 4.1: A particularly neat example of an Einstein Ring from https://upload.wikimedia.org/wikipedia/commons/1/11/A_Horseshoe_Einstein_Ring_from_Hubble.jpg. The image (from Hubble Deep Field) of a distant blue galaxy is distorted into a ring by the gravitational field around a closer Luminous Red Galaxy.

We therefore turn our attention to null geodesics in the Levi-Civita spacetime. For reference, we were able to write the metric in the following form:

\[ ds^2 = -\left(\frac{\rho}{\rho_0}\right)^{4\lambda} dt^2 + \left(\frac{\rho}{\rho_0}\right)^{-4\lambda(1-2\lambda)} d\rho^2 + \left(\frac{\rho}{\rho_0}\right)^{-4\lambda(1-2\lambda)} dz^2 + D^2 \left(\frac{\rho}{\rho_0}\right)^{2-4\lambda} \rho_0^2 d\phi^2, \]

(4.1)

where \(\rho_0\) is some arbitrary distance, and \(\lambda\) is a parameter related to the filament’s mass per unit length. The Lagrangian for a photon can be written

\[ \mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \]

(4.2)

where dots refer to derivatives with respect to "affine parameter", an analogue of proper time along the worldline of a photon. We expanded this for a photon traveling in the Levi-Civita...
Because this Lagrangian does not depend on $t$, $z$, or $\phi$, we can construct a conserved quantity for each of these variables:

\[
\frac{\partial \mathcal{L}}{\partial \dot{t}} = -E = -\left(\frac{\rho}{\rho_0}\right)^{4\lambda} \dot{t}
\]

\[
\frac{\partial \mathcal{L}}{\partial \dot{z}} = p_z = \left(\frac{\rho}{\rho_0}\right)^{-4\lambda(1-2\lambda)} \dot{z}
\]

\[
\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = J = D^2 \rho^2 \left(\frac{\rho}{\rho_0}\right)^{-4\lambda} \dot{\phi}
\]

We derive a differential equation for each component of the photon trajectory:

\[
\dot{t} = \left(\frac{\rho}{\rho_0}\right)^{-4\lambda} E
\]

\[
\dot{z} = \left(\frac{\rho}{\rho_0}\right)^{4\lambda(1-2\lambda)} p_z
\]

\[
\dot{\phi} = D^{-2} \rho^{-2} \left(\frac{\rho}{\rho_0}\right)^{4\lambda} J
\]

Lastly, we need an equation for the behavior of $\rho$. This could be derived from the Euler-Lagrange equation for $\rho$, but it is more convenient to consider the definition of a null geodesic,

\[
g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0 = -\left(\frac{\rho}{\rho_0}\right)^{4\lambda} \dot{t}^2 + \left(\frac{\rho}{\rho_0}\right)^{-4\lambda(1-2\lambda)} \dot{\rho}^2 + D^2 \rho^2 \left(\frac{\rho}{\rho_0}\right)^{-4\lambda} \dot{\phi}^2 + \left(\frac{\rho}{\rho_0}\right)^{-4\lambda(1-2\lambda)} \dot{z}^2
\]
4.1. FEMTOLENSING IN THE LEVI-CIVITA SPACETIME

We can substitute in our conserved quantities in order to eliminate the dependence on other variables, leaving us with

\[- \left( \frac{\rho}{\rho_0} \right)^{-4\lambda} E^2 + \left( \frac{\rho}{\rho_0} \right)^{-4\lambda(1-2\lambda)} \dot{\rho}^2 + D^{-2} \rho^{-2} \left( \frac{\rho}{\rho_0} \right)^{+4\lambda} J^2 + \left( \frac{\rho}{\rho_0} \right)^{+4\lambda(1-2\lambda)} p_z^2 = 0. \quad (4.11)\]

We can multiply both sides by \( \left( \frac{\rho}{\rho_0} \right)^{+4\lambda(1-2\lambda)} \) and we are left with

\[- \left( \frac{\rho}{\rho_0} \right)^{-8\lambda^2} E^2 + \rho^2 + D^{-2} \rho^{-2} \left( \frac{\rho}{\rho_0} \right)^{8\lambda-8\lambda^2} J^2 + \left( \frac{\rho}{\rho_0} \right)^{8\lambda(1-2\lambda)} p_z^2 = 0. \quad (4.12)\]

Solving for \( \dot{\rho} \), we find

\[\dot{\rho} = \pm \sqrt{\left( \frac{\rho}{\rho_0} \right)^{-8\lambda^2} E^2 - D^{-2} \rho^{-2} \left( \frac{\rho}{\rho_0} \right)^{8\lambda-8\lambda^2} J^2 + \left( \frac{\rho}{\rho_0} \right)^{8\lambda(1-2\lambda)} p_z^2} \quad (4.13)\]

\[= \pm E \left( \frac{\rho}{\rho_0} \right)^{-4\lambda^2} \sqrt{1 - D^{-2} \rho^{-2} \left( \frac{\rho}{\rho_0} \right)^{8\lambda} j^2 + \left( \frac{\rho}{\rho_0} \right)^{8\lambda-8\lambda} \pi_z^2} \quad (4.14)\]

where we have defined the scaled quantities \( j := \frac{j}{E} \) and \( \pi_z := \frac{p_z}{E} \). From this expression, we can make a number for observations. First off, the negative solution will correspond to the path approaching the filament, and the positive solution will correspond to the photon moving away from the filament. We expect there to be some turning point, where the photon makes its closest approach to the filament, a point where \( \dot{\rho} = 0 \). For simplicity, let us consider the case were \( \pi_z = 0 \), which is to say that the photon has no momentum along
the $z$ axis. We can easily solve for the critical value for $\rho$, which we will call $\rho_c$:

$$0 = 1 - D^{-2} \rho_c^{-2} \left( \frac{\rho_c}{\rho_0} \right)^{8\lambda} j^2$$

(4.15)

$$1 = \frac{j^2}{D^2} \left( \frac{\rho_c^{8\lambda-2}}{\rho_0^{8\lambda}} \right)$$

(4.16)

$$\rho_c = \rho_0 \left( \frac{j}{\rho_0 D} \right)^{\frac{1}{\lambda}}$$

(4.17)

Though it is more difficult to solve for $\rho_c$ in the more general case of $\pi_z \neq 0$, it can be done via numerical methods in principle.

Equipped with this turning point, we can now write down an expression for the total angle of deflection relative to the photon path:

$$\Delta \phi = \int_{\rho_i}^{\rho_c} \frac{\dot{\phi}}{\dot{\rho}} d\rho + \int_{\rho_c}^{\rho_f} \frac{\dot{\phi}}{\dot{\rho}} d\rho$$

(4.18)

where $\rho_i$ is the initial $\rho$ coordinate distance from the filament, and $\rho_f$ is the corresponding final $\rho$ value where the photons are to converge and be detected. We can define an integral function to simplify this,

$$I_\phi(a, b, j, \pi_z) = \int_a^b \frac{\dot{\phi}}{\dot{\rho}} d\rho$$

(4.19)

$$= \int_a^b \frac{D^{-2} \rho^{-2} \left( \frac{\rho}{\rho_0} \right)^{4\lambda} J}{\pm E \left( \frac{\rho}{\rho_0} \right)^{-4\lambda^2} \sqrt{1 - D^{-2} \rho^{-2} \left( \frac{\rho}{\rho_0} \right)^{8\lambda} j^2 + \left( \frac{\rho}{\rho_0} \right)^{8\lambda-8\lambda} \pi_z^2}} d\rho$$

(4.20)

$$= \int_a^b \frac{D^{-2} \rho^{-2} \left( \frac{\rho}{\rho_0} \right)^{4\lambda+4\lambda^2} j}{\pm \sqrt{1 - D^{-2} \rho^{-2} \left( \frac{\rho}{\rho_0} \right)^{8\lambda} j^2 + \left( \frac{\rho}{\rho_0} \right)^{8\lambda-8\lambda} \pi_z^2}} d\rho$$

(4.21)

where the sign of the denominator is determined by whether the ray is moving towards or
4.1. FEMTOLENSING IN THE LEVI-CIVITA SPACETIME

Away from the filament. We can now express the net deflection as

$$\Delta \phi = I_\phi(\rho_i, \rho_c, j, \pi_z) + I_\phi(\rho_c, \rho_f, j, \pi_z) \quad (4.22)$$

We can derive a similar expression for the coordinate time elapsed along the photon orbit, which will ultimately allow us to calculate the phase difference and furthermore the interference pattern.

$$\Delta t = \int_{\rho_i}^{\rho_c} \frac{i}{\rho} d\rho + \int_{\rho_c}^{\rho_f} \frac{i}{\rho} d\rho \quad (4.23)$$

where the limits of integration are identical to the ones used in the $\phi$ integrals. Once again, we can define an integral function

$$I_t(a, b, j, \pi_z) = \int_a^b \frac{i}{\rho} d\rho \quad (4.24)$$

such that

$$\Delta t = I_t(\rho_i, \rho_c, j, \pi_z) + I_t(\rho_c, \rho_f, j, \pi_z) \quad (4.27)$$

We now have the tools we need to calculate the interference pattern, and all that is left is to piece them together.
CHAPTER 4. GRAVITATIONAL LENSING AND FEMTOLENSING

4.2 A “Simple” Example

In order to illustrate this effect, we turn our attention to the case where $\pi_z = 0$, $D = 1$, and $\rho_i = \rho_f = \rho_0$. Note that this final condition assumes that the filament is halfway between the light source and the observer, a condition that simplifies further arguments, and removes the need for accounting for gravitational redshifting (whatever occurs on the way ”in” is undone on the way ”out”). This condition can be lifted, however, without adding much complexity to the problem. The condition that $\rho_f = \rho_0$ is NOT a physical restriction; it simply amounts to making a choice for the arbitrary parameter $\rho_0$, analogous to choosing a constant to add to an electrostatic potential. This situation is illustrated in figure 4.2.

Based on our intuition, we expect there to be two rays of light that can reach any given point of detection: one that travels above the filament and one that travels below. This intuition has one subtlety: it can be shown that as the mass parameter $\lambda$ approaches its allowable limit of $1/4$, it becomes possible for a third and even a fourth ray to be lensed between the same two locations. We neglect this complication here, mainly because we are more interested in the more astrophysically-plausible case where $\lambda$ has a value much less than unity. We can call the detector’s angular displacement from the axis (see figure 4.2)
some small value $\epsilon$ and claim that the paths must undergo a deflection of $\Delta \phi = \pi + \epsilon$ and $\Delta \phi = -\pi + \epsilon$ respectively. The integral expressions for $\Delta \phi$ now become

\[
\Delta \phi_P = \int_{\rho_0}^{\rho_c} \rho^{-2} \left( \frac{\rho}{\rho_0} \right)^{4\lambda + 4\lambda^2} \frac{j}{\sqrt{1 - \rho^{-2} \left( \frac{\rho}{\rho_0} \right)^{8\lambda} j^2}} d\rho + \int_{\rho_0}^{\rho_c} \rho^{-2} \left( \frac{\rho}{\rho_0} \right)^{4\lambda + 4\lambda^2} \frac{j}{-\sqrt{1 - \rho^{-2} \left( \frac{\rho}{\rho_0} \right)^{8\lambda} j^2}} d\rho = \pi + \epsilon \tag{4.28}
\]

\[
\Delta \phi_N = \int_{\rho_0}^{\rho_c} \rho^{-2} \left( \frac{\rho}{\rho_0} \right)^{4\lambda + 4\lambda^2} \frac{j}{\sqrt{1 - \rho^{-2} \left( \frac{\rho}{\rho_0} \right)^{8\lambda} j^2}} d\rho + \int_{\rho_0}^{\rho_c} \rho^{-2} \left( \frac{\rho}{\rho_0} \right)^{4\lambda + 4\lambda^2} \frac{j}{-\sqrt{1 - \rho^{-2} \left( \frac{\rho}{\rho_0} \right)^{8\lambda} j^2}} d\rho = -\pi + \epsilon \tag{4.29}
\]

where $\Delta \phi_P$ and $\Delta \phi_N$ are the deflection angles for the paths above and below the filament respectively. Because of our choices for the detector and source $\rho$ values, we can reverse the integral limits, at the cost of a minus sign, and combine the terms, leaving

\[
\Delta \phi_P = \int_{\rho_0}^{\rho_c} 2\rho^{-2} \left( \frac{\rho}{\rho_0} \right)^{4\lambda + 4\lambda^2} \frac{j}{\sqrt{1 - \rho^{-2} \left( \frac{\rho}{\rho_0} \right)^{8\lambda} j^2}} d\rho = \pi + \epsilon \tag{4.30}
\]

\[
\Delta \phi_N = \int_{\rho_0}^{\rho_c} 2\rho^{-2} \left( \frac{\rho}{\rho_0} \right)^{4\lambda + 4\lambda^2} \frac{j}{\sqrt{1 - \rho^{-2} \left( \frac{\rho}{\rho_0} \right)^{8\lambda} j^2}} d\rho = -\pi + \epsilon. \tag{4.31}
\]

If intuition is to be trusted, we expect there to be one value of $j$ that will satisfy each of these equations. The solutions will be presented in the next section along with our other results.

For each value of $j$ we must now calculate $\Delta t$ in order to find the phase difference. For some $j$, in this simple case, we have

\[
\Delta t(j) = \int_{\rho_0}^{\rho_c} \left( \frac{\rho}{\rho_0} \right)^{-4\lambda + 4\lambda^2} \frac{d\rho}{\sqrt{1 - \rho^{-2} \left( \frac{\rho}{\rho_0} \right)^{8\lambda} j^2}} + \int_{\rho_0}^{\rho_c} \left( \frac{\rho}{\rho_0} \right)^{-4\lambda + 4\lambda^2} \frac{d\rho}{-\sqrt{1 - \rho^{-2} \left( \frac{\rho}{\rho_0} \right)^{8\lambda} j^2}}. \tag{4.32}
\]
It can once again be noted that by reversing the bounds of integration we can combine the integrals, leaving

\[
\Delta t(j) = \int_{\rho_0}^{\rho} \frac{2\left(\frac{\rho}{\rho_0}\right)^{-4\lambda+4\lambda^2}}{\sqrt{1 - \rho^{-2}\left(\frac{\rho}{\rho_0}\right)^{8\lambda} j^2}} d\rho.
\] (4.33)

The difference in travel time for the two beams, \(\Delta T\), can now be written down as

\[
\Delta T(\epsilon) = \Delta t(j_P[\epsilon]) - \Delta t(j_N[\epsilon])
\] (4.34)

where \(j_P[\epsilon]\) and \(j_N[\epsilon]\) are the \(j\) values that satisfy equations 4.30 and 4.31 respectively for a given value of \(\epsilon\).

The time averaged intensity of an interference pattern between two beams is proportional to the \(\cos^2\) of the phase difference between them, so we can write

\[
\langle I \rangle(\epsilon, \omega) \propto \cos^2\left(\frac{\omega \Delta T(\epsilon)}{2}\right)
\] (4.35)

\[
\langle I \rangle(\epsilon, \omega) \propto \cos^2\left(\frac{\omega (\Delta t(j_P[\epsilon]) - \Delta t(j_N[\epsilon]))}{2}\right)
\] (4.36)

where \(\omega\) is the frequency of light. This can be shown by considering the field amplitude:

\[
E \propto \cos(\omega t) + \cos(\omega t + \phi).
\] (4.37)

Applying trigonometric identities, this can be written

\[
E \propto \cos(\omega t) + \cos(\omega t)\cos(\phi) - \sin(\omega t)\sin(\phi).
\] (4.38)
4.2. A “SIMPLE” EXAMPLE

When this expression is squared to get the intensity,

\[
I \propto \cos^2(\omega t) + \cos^2(\omega t) \cos^2(\phi) + \sin^2(\omega t) \sin^2(\phi) \\
+ 2 \cos^2(\omega t) \cos(\phi) - 2 \cos(\omega t) \sin(\omega t) \sin(\phi) - \cos(\omega t) \sin(\omega t) \cos(\phi) \sin(\phi)
\]  
(4.39)

When this intensity is time averaged, we are left with

\[
\langle I \rangle \propto 1/2 + (1/2) \cos^2(\phi) + (1/2) \sin^2(\phi) + \cos(\phi)
\]  
(4.40)

\[
\propto 1 + \cos(\phi)
\]  
(4.41)

\[
\propto 2 \cos^2(\phi/2)
\]  
(4.42)

leaving us with our desired dependence on \( \phi \). So now, assuming we can solve for \( j_P[\epsilon] \) and \( j_N[\epsilon] \), and also solve the integral in equation 4.33 we can in principle plot the interference pattern generated by the Levi-Civita metric for a given linear mass density \( \lambda \). This is complementary to previous work [I], which has shown that in the case of cosmic strings with \( \lambda = 0 \), but with \( D \neq 0 \), a similar procedure can be used to calculate \( D \) from the femtolensing pattern.

When we plot this averaged intensity for a given \( \lambda \) as a function of the angular displacement \( \epsilon \) and the frequency of light \( \omega \), we find an interference pattern, displayed in figure 4.3.

We can analyze the features of this plot more easily by considering the first minu-
Figure 4.3: A plot of the interference pattern generated by femtolensing in the Levi-Civita metric with $\lambda = 0.15$ and $\rho_i = \rho_f = \rho_0 = 1$. The color weighting corresponds to the relative intensity of light, given a value for the angular displacement $\epsilon$ (plotted on the horizontal axis in radians) and the frequency of light $\omega$ (plotted on the vertical axis in dimensionless units $\omega = f \ast \rho_0 / c$).
mum fringe in the $\omega - \epsilon$ space. This curve is given by

$$\omega(\epsilon) = \frac{2\pi}{\Delta T(\epsilon)}$$  \hspace{1cm} (4.43)

We plotted this first interference fringe for a variety of values for $\lambda$, and these results are displayed in figure 4.4. These curves appear to adhere closely to a simple power law of the form

$$\omega(\epsilon) = A(\lambda)\epsilon^{P(\lambda)}$$  \hspace{1cm} (4.44)

where $A(\lambda)$ and $P(\lambda)$ are simple functions of the linear mass density. The parameter $A$ seems to vary much more significantly with respect to changes in $\lambda$ than $P(\lambda)$, which stays close to a value of -1. This behavior is especially apparent in a log-log scale plot of this relationship, as displayed in figure 4.5.

Additionally, we can plot the variation in these coefficients with respect to $\lambda$, and these results are presented in figures 4.6 and 4.7 respectively. The behavior of these plots makes intuitive sense with one exception: we are still unclear as to how to interpret the increase of these functions for large $\lambda$. This may be due to $\lambda$ approaching it’s a maximum value, or some numerical error, but at this point the intuitive explanation is still unclear.

Regardless, it is clear that if such a signal were detected, it would in principle, though perhaps not so practically, be possible to infer the mass density of the string from the interference pattern. Further work would be able to determine the plausibility of this method for determining $\lambda$ for more general and more complicated variations of this problem, for instance where $\rho_i \neq \rho_f$ and $\pi_z \neq 0$. This much, however, has yet to be shown.
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Figure 4.4: A plot of the first minimum fringe for various values of the linear mass density $\lambda$. Each curve is generated by calculating $\omega(\epsilon) = \frac{2\pi}{\Delta T(\epsilon)}$. The curves appear to follow a power law of the form $\omega(\epsilon) = A\epsilon^{-P}$ where $P$ and $A$ are coefficients that depend on $\lambda$ in some non-trivial way.
4.2. A “SIMPLE” EXAMPLE

Figure 4.5: A log-log scale plot of the first fringe curves from figure 4.4. Once again, the shift parameter $A$’s dependence on $\lambda$ is very apparent. The exponent $P$ is not consistent between the curves, but its variation is not very clear in this figure. See figure 4.6 for a more careful look at the exponent’s dependence on $\lambda$. 
Figure 4.6: This figure shows the dependence of $P$ on the linear mass density $\lambda$. For small $\lambda$, the sharp increase in the exponent makes sense since we expect the interference pattern to diverge (in both $P$ and $A$) for small enough $\lambda$, as the lensing effects should vanish. It is worth noting that the deviations in the exponent function $P(\lambda)$ are much smaller than the deviations in $A(\lambda)$. 
4.2. A “SIMPLE” EXAMPLE

Figure 4.7: This figure shows the dependence of $P$ on the linear mass density $\lambda$. For small $\lambda$, the sharp increase in the exponent makes sense since we expect the interference pattern to diverge (in both $P$ and $A$) for large enough $\lambda$, as the lensing effects should vanish. The increase in the $A$ as $\lambda$ approaches .2 is currently lacking an intuitive explanation.
Chapter 5

Fluid Atmosphere Near a Massive Filament

The Levi-Civita solution is a vacuum solution to Einstein’s equations. This means that there is no matter in the spacetime other than the singular filament at $\rho = 0$. Matter can still accumulate around the filament, so long as the matter does “self gravitate”, meaning it must not contribute significantly to the gravitational field, in which case the Levi-Civita solution would become invalid and we would have to resolve a more complicated set of Einstein-Fluid equations that provide spacetimes compatible with self gravitating fluids. Including this back reaction of the fluid would certainly be an interesting result, but in this thesis we focus on the case where the back reaction of the fluid on the metric can be ignored. In this section, we will explore how the laws of hydrodynamics describe an accretion cloud of fluid in the Levi-Civita spacetime. This technique of studying the accretion of matter onto
a stationary metric has been very well developed in the context of stellar and black-hole spacetimes, and the framework is readily applicable to the Levi-Civita metric.

5.1 Hydrodynamics in General Relativity

We will now turn our attention to the study of a fluid’s motion in a relativistic framework. The following techniques are very well established and there are countless resources available which explain the theory with great rigor. As such, this discussion will largely be a summary. For further reading on the framework of relativistic hydrodynamics, I would point the reader to the wealth of material available, especially [9] and [10].

In relativistic hydrodynamics, the goal is to model the fluid rest mass density \( \rho \) (the baryonic number density \( n \) times the mass of these baryons), the internal energy \( \epsilon \), the pressure \( P \), and the fluid four-velocity \( U^\mu = \frac{dx^\mu}{d\tau} \), where \( \tau \) represents the proper time measured in the rest frame of the fluid. Of these, \( \epsilon, \rho, \) and \( P \) are defined in the fluid element’s rest frame. Note that since we are now using \( \rho \) to denote mass density, we need a new letter to denote cylindrical radius. From this point on, we will refer to the this coordinate as \( r \).

There are a number of simplifications that can be made before even considering the metric.

First of all, it is easily shown that \( U^\mu \), under the constraint that the fluid can only travel timelike paths, must be normalized by the condition

\[
U_\nu U^\nu = g_{\mu\nu} U^\mu U^\nu = -1. \tag{5.1}
\]

This is to say, if we can find three of the components, we can determine the last trivially.
5.1. HYDRODYNAMICS IN GENERAL RELATIVITY

This comes from the definition of $U^\mu$:

$$U^\mu = \frac{dx^\mu}{d\tau} \quad (5.2)$$

Where $\tau$ is the proper time. So when we take the inner product using the metric $g_{\mu\nu}$, we have

$$U_\mu U^\mu = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx_\nu}{d\tau} = \frac{g_{\mu\nu} dx^\mu dx_\nu}{d\tau^2} \quad (5.3)$$

but the numerator is just $-d\tau^2$ for a timelike path, so we are left with our desired condition, $U_\mu U^\mu = -1$.

For the sake of simplicity, we will also consider the case of a perfect fluid, where the pressure is defined by the perfect fluid equation of state,

$$P = \rho e (\Gamma - 1), \quad (5.4)$$

where $\Gamma$ is an adiabatic index, which we take to be constant. It is also useful to consider the relativistic enthalpy density, defined as

$$h = 1 + \epsilon + \frac{P}{\rho} \quad (5.5)$$

The laws of fluid dynamics are described by two conservation laws, conservation of stress-energy and conservation of baryon number. The law of Baryon conservation is written

$$\nabla^\mu (\rho U_\mu) = 0 \quad (5.6)$$
and the conservation of stress energy is written

\[ \nabla_\mu T^{\mu \nu} = 0 \]  \hspace{1cm} (5.7)

It is common to decompose this equation into how it would be understood by an observer moving along with the fluid (that is, with four-velocity \( \vec{U} \)). To such an observer, the timelike component of this equation represents the conservation of energy:

\[ U^\nu \nabla_\nu T_{\mu \nu} = 0, \]  \hspace{1cm} (5.8)

and the spacelike components, factored out with the help of a mathematical object called the projection operator \( P_{\mu \nu} = g_{\mu \nu} + U_\mu U_\nu \), represent the conservation of momentum:

\[ (g_{\gamma \mu} + U_\gamma U_\mu) \nabla_\nu T^{\mu \nu} = 0. \]  \hspace{1cm} (5.9)

For a perfect fluid, the stress energy tensor takes the form

\[ T^{\mu \nu} = \rho h U^\mu U^\nu + P g^{\mu \nu} \]  \hspace{1cm} (5.10)

where \( \rho \) is the baryonic mass density, \( h \) is the relativistic enthalpy density (defined by [5.5]), \( U^\mu \) is the fluid four-velocity, and \( P \) is the pressure.
There are an overwhelming number of fluid variables that we could choose to keep track of as a function of our chosen coordinates. The most physically interesting quantity is probably the fluid density $\rho$. The goal here is therefore to isolate the density of fluid $\rho$ as a function of our cylindrical coordinates $r$,$\phi$,$z$, and $t$. The baryon density conservation equation is a logical place to start due to its simplicity. Using a convenient trick to eliminate the covariant derivative, we can rewrite

$$\nabla^\mu (\rho U_\mu) = 0$$ \hspace{1cm} (5.11)

$$\frac{1}{\sqrt{|g|}} \partial_\mu (\rho \sqrt{|g|} U^\mu) = 0,$$ \hspace{1cm} (5.12)

where $|g|$ is the absolute value of the determinant of the metric $g_{\mu\nu}$. If we consider our system to have cylindrical symmetry, then we expect all derivatives to go to zero except for the $r$ derivatives, which leaves us with

$$\rho \sqrt{|g|} U^r = \text{constant} = k_\rho.$$ \hspace{1cm} (5.13)

$$\rho = \frac{k_\rho}{\sqrt{|g|} U^r}.$$ \hspace{1cm} (5.14)

That’s a good start; two of the relevant fluid variables are now related in a simple way. Now we turn to the conservation of energy. First off, if we plug our stress-energy components into the energy conservation equation, we have

$$U^\nu \left( \nabla^\mu [\rho h U_\mu U_\nu] + \nabla_\nu P \right) = 0$$ \hspace{1cm} (5.15)
If we are looking for equilibrium solutions, where all derivatives with respect to \( t \) are zero, it can be shown \([11]\) that the energy conservation equation reduces to

\[
U^{\mu} (\nabla^{\mu} [\rho h U_{\mu} U_{\nu}] + \nabla_{\nu} P) = 0
\]

and then invoking the Leibniz rule and making more substitutions, including baryon conservation to eliminate the first term,

\[
\epsilon \partial_i (\sqrt{|g|} \rho U^i) + \sqrt{|g|} \rho U^i \partial_i (\epsilon) + \rho \epsilon (\Gamma - 1) \partial_i (\sqrt{|g|} U^i) = 0
\]

Now we’re close. We can manipulate the equation using the chain rule for logarithms to progress:

\[
\frac{1}{\epsilon} \partial_i (\epsilon) = \frac{1 - \Gamma}{\sqrt{|g|} U^i} \partial_i (\sqrt{|g|} U^i)
\]

\[
\partial_i (\log(\epsilon)) = \partial_i ((1 - \Gamma) \log(\sqrt{|g|} U^i))
\]

\[
\epsilon = k' (\sqrt{|g|} U^i)^{1-\Gamma}
\]

\[
\epsilon = k' \left( \frac{k_B}{\rho} \right)^{1-\Gamma}
\]

\[
\epsilon = k_\epsilon \rho^{\Gamma-1}.
\]

We now have three of the relevant fluid variables, \( \rho, U^r, \) and \( \epsilon \) related. We will now use the conservation of momentum to produce a differential equation for \( \rho \) which can be solved numerically. It is worth remembering that we could have just as easily used these results to
pursue a differential equation for $U^r$ or $\epsilon$.

When we plug the perfect fluid stress energy tensor into the momentum conservation equation, we have

$$\nabla_\mu (\rho h U^\mu U_\nu + P g^\mu_\nu) = 0.$$  

(5.25)

We can use the Leibniz rule to arrive at

$$h U_\nu \nabla_\mu (\rho U^\mu) + \rho U^\mu \nabla_\mu (h U_\nu) + \nabla_\mu (P g^\mu_\nu) = 0$$  

(5.26)

$$\rho U^\mu \nabla_\mu (h U_\nu) + \partial_\nu P = 0$$  

(5.27)

where we have invoked baryon conservation to eliminate the first term and the fact that $g^\mu_\nu = \delta^\mu_\nu$ to simplify the last. We can now expand the covariant derivative and make a substitution suggested by Hawley, Smarr, and Wilson [11] to receive

$$\rho U^\mu (\partial_\mu (h U_\nu) - \Gamma^\alpha_{\mu\nu} h U_\alpha) + \partial_\nu P = 0$$  

(5.28)

$$\rho U^\mu \partial_\mu (h U_\nu) + \frac{1}{2} \rho h U_\alpha U_\mu \partial_\nu g^\alpha_\mu + \partial_\nu P = 0.$$  

(5.29)

We now have one differential equation of each choice of $\nu$. Thankfully, three of them result in straightforward results when we assume an axisymmetric stationary solution independent of any coordinate but $r$. For a non-$r$ choice of $\nu$, designated $\nu = A$, we are discover

$$\rho U^r \partial_r (h U_A) = 0$$  

(5.30)
which results in convenient expressions for most of the fluid four-velocity components:

\[ U_t = \frac{k_t}{h} \quad (5.31) \]
\[ U_\phi = \frac{k_\phi}{h} \quad (5.32) \]
\[ U_z = \frac{k_z}{h} \quad (5.33) \]

where \( k_t, k_\phi \), and \( k_z \) are constants that can be determined via the normalization of the four velocity components, along with the result of baryon conservation.

Now all that is left is the differential equation for \( \nu = r \):

\[ \rho U^r \partial_r (h U_r) + \frac{1}{2} \rho h U_\alpha U_\mu g^{\alpha \mu} + \partial_r P = 0. \quad (5.34) \]

It is now convenient to expand and substitute in our expression for relativistic enthalpy, \( h \).

We know that

\[ h = \epsilon \Gamma + 1 = (k_\epsilon \rho^{\Gamma - 1}) + 1, \quad (5.35) \]

so we can now plug in for our differential equation for \( \rho \) and \( U_\mu \),

\[ \rho U^r \partial_r (k_\epsilon \rho^{\Gamma - 1} U_r + U_r) + \partial_r ((\Gamma - 1)k_\epsilon \rho^\Gamma) + \frac{1}{2}(\rho + k_\epsilon \rho^\Gamma)U_\alpha U_\mu \partial_\nu g^{\alpha \mu} = 0. \quad (5.36) \]
5.3. HYDRODYNAMICS RESULTS

This differential equation can be combined with our other results for \( U_\mu \):

\[
\frac{k_\rho}{\sqrt{|g|}} \partial_r (k_\rho \rho^{-2 \Gamma} g^{rr}) + \frac{k_\rho}{\sqrt{|g|} g^{\Gamma \rho}} + \partial_r \left( (\Gamma - 1) k_\rho \rho^\Gamma \right) \\
+ \frac{1}{2} \left( \rho + k_\rho \rho^\Gamma \right) \left[ \left( \frac{k_\rho}{g^{rr} \sqrt{|g|}} \right)^2 \partial_r g^{rr} + \left( \frac{k_\rho}{(k_\rho \rho^{\Gamma - 1}) + 1} \right)^2 \partial_r g^{tt} \\
+ \left( \frac{k_z}{(k_\rho \rho^{\Gamma - 1}) + 1} \right)^2 \partial_r g^{zz} + \left( \frac{k_\phi}{(k_\rho \rho^{\Gamma - 1}) + 1} \right)^2 \partial_r g^{\psi \psi} \right] = 0. \quad (5.37)
\]

As desired, we have arrived at a differential equation solely relating \( \rho \) and \( r \) for the axisymmetric stationary case. This same process could have in principle been accomplished while making fewer assumptions, but this simple case turns out to reveal some interesting accretion phenomena for the Levi-Civita spacetime.

5.3 Hydrodynamics Results

There are a variety of parameters that can be tuned in order to search for interesting solutions to these equations. One solution of particular significance is the case where \( k_\rho = k_z = 0, k_\phi \neq 0 \) and \( k_t \neq 0 \), corresponding to a stable cylindrical cloud of dust surrounding the central axis. The results for this case are presented in figure 5.1.

When we consider the important features of this particular set of solutions, we should note the physical implication of setting \( k_\rho = 0 \). By equation 5.14, it is clear that by setting \( k_\rho = 0 \), we imply that \( U^r = U_r = 0 \). The fluid four velocity in this case has no component pointing in the radial direction, so the fluid is not collapsing towards the filament. Similarly, we have set \( k_z = 0 \) which, using equation 5.33, implies that \( U^z = U_z = 0 \), so the
Figure 5.1: A plot of density of dust as a function of the radial coordinate $r$ in a Levi-Civita spacetime for various initial conditions. Each curve was generated by solving the hydrodynamic equations 5.37 in the case where $\lambda = 0.1$, $k_\rho = k_z = 0$, $k_\phi \neq 0$ and $k_t \neq 0$, corresponding to a stable cylindrical cloud of dust surrounding the central axis.

The fluid is not traveling along the filament in the $z$ direction either. On the other hand, because $k_\phi \neq 0$ and $k_t \neq 0$, by equation 5.33 we know the four velocity components in the $\phi$ and $t$ directions are nonzero. So the fluid is rotating around the central axis.

The particular set of solutions presented in figure 5.1 represent cylindrically symmetric distributions of fluid density held up by their angular momentum at a fixed radial distance from the filament. These solutions are stationary: the fluid density distributions do not change in time. We made a lot of simplifying assumptions to arrive at this result, but the principles used could be just as readily used to determine more complicated fluid density distributions such as ones where more components of the four velocity are nonzero, or ones that fluctuate with time, but this is left for future work.
Chapter 6

Conclusion

Spacetimes with cylindrical symmetry, while currently having few astrophysical applications, remain a consistent source of mathematical and intuitive subtlety in the world of general relativity. We have shown that the linear mass density of a cosmic string of massive filament can in principle be determined from the femtolensing interference signature characteristic of a given geometry. Despite the subtleties in coordinates and dimensional units, solutions to the hydrodynamic equations of equilibrium can be found to model the accretions around an object such as a cosmic string or massive filament corresponding to the Levi-Civita spacetime. This said, many more subtleties remain to be explored. Additionally, the behavior of the fringe positions for Levi-Civita spacetimes with small $\lambda$ makes intuitive sense, but an intuitive gap remains in understanding the large $\lambda$ behavior. The tools and procedures outlined in this thesis could be used to further explore these subtleties and the subtleties of other spacetimes with cylindrical symmetry, such as the van Stockum Dust solution. The equations of hydrodynamics presented in the last chapter could potentially be used to un-
cover a wealth of diverse fluid solutions, and a clear next step would be to include the back reaction of a weakly self gravitating fluid onto the Levi-Civita metric.
Chapter 7

Appendix: Mathematical Background

A thorough understanding of theory of general relativity is built upon a vast underpinning of mathematical formalism. This thesis was intended to briefly outline a skeleton of the material that could have been explored, exposing the practical outcomes of the theory without addressing the mathematical machinery used along the way. There are countless texts available that explore richness and complexity of this machinery. Having said this, I figured a concise summary of some of the more important foundational machinery might be of use to a reader who is otherwise unfamiliar with the concepts. For a much more rigorous approach, I highly recommend Sean Carroll [4] and Misner, Thorne, and Wheeler [5], for I have found their analysis invaluable, and have referred to them extensively in my writing of this appendix.
7.1 Vectors, Covectors, and Tensors

One topic notorious for plaguing introductions to general relativity is the generalized formalism for vectors. Introductory students of physics typically hold their Euclidean intuition for vectors very close to their hearts. According to this physical intuition, vectors are arrows, starting at one point specified by some coordinates and pointing in some direction towards and ending at another point specified by some other coordinates. This is how vector mechanics is taught in introductory courses, sometimes even using a home made meter stick “arrow.” This understanding is, however, anathema to relativists.

First of all, vectors are not objects that span “distance” between points as suggested by the meter stick school of thought. On a manifold, vectors (also specified as contravariant vectors) are defined at a point. At each individual point on a manifold, the set of all vectors is defined in a vector space called the tangent space, named according to the “plane” generated by the set of all vectors tangent to a point in curved two-dimensional space. While this is a useful tool, the analogy is a bit harder to picture in the three or four dimensional cases. As mentioned in the note on notation at the beginning of this thesis, contravariant vectors are represented in terms of some basis $\hat{e}_\mu$ as

$$\vec{V} = V^\mu \hat{e}_\mu$$  \hspace{1cm} (7.1)

where $V^\mu$ are the components of the vector in that particular basis.

So now there is some ambiguity. We want the basis vectors to form a coordinate basis for all the vectors in the tangent space, but in curved spacetime, it’s not inherently obvious which vectors these should be, since our intuition for geometry is out the window.
It turns out that the coordinate basis vectors we are looking for are the partial derivative operators \( \partial_\mu \) for some choice of coordinates \( x^\mu \). It follows that the transformation law for these basis vectors into some new coordinate system \( x'^\mu \) is written

\[
\partial_\nu' = \frac{\partial x^\mu}{\partial x'^\mu} \partial_\mu. \tag{7.2}
\]

Then, if the complete vector \( \vec{V} = V^\mu \partial_\mu \) is going to be coordinate independent, it must follow that the transformation law for the vector components \( V^\mu \) is

\[
V'^\nu = \frac{\partial x'^\nu}{\partial x^\mu} V^\mu, \tag{7.3}
\]

such that

\[
\vec{V}' = V'^\nu \partial_\nu' = \frac{\partial x'^\nu}{\partial x^\mu} V^\mu \frac{\partial x^\mu}{\partial x'^\mu} \partial_\mu = V^\mu \partial_\mu. \tag{7.4}
\]

Given this “contravariant” tangent space, we can define another vector space, known as the cotangent space, the set of all covectors at that point. Now, there’s a lot of names for covectors: one-forms, dual-vectors, covariant vectors. All of these names refer to objects that act as linear maps from the vectors to real numbers. The set of all these objects constitutes a vector space of its own. From a set of basis vectors, we can construct a set of covariant basis vectors, requiring that the function of those covariant basis vectors on the contravariant basis vectors is the Kronecker delta. This is to say,

\[
\hat{\theta}^\nu (\hat{e}_\mu) = \delta_\mu^\nu \tag{7.5}
\]

Covectors can be defined in terms of these basis vectors similarly (but not identically!) to
CHAPTER 7. APPENDIX: MATHEMATICAL BACKGROUND

contravariant vectors, by specifying components. A covector \( \bar{\omega} \) is written

\[
\bar{\omega} = \omega_{\mu} \hat{\theta}^\mu. \tag{7.6}
\]

where \( \omega_{\mu} \) are the covariant components. Knowing this, the action of covectors on vectors is easily derived!

\[
\bar{\omega} \left( \vec{V} \right) = \omega_{\mu} \hat{\theta}^\mu V^\nu \hat{e}_\nu \tag{7.7}
\]

\[
= \omega_{\mu} V^\nu \hat{\theta}^\mu \hat{e}_\nu \tag{7.8}
\]

\[
= \omega_{\mu} V^\nu \delta^\mu_\nu \tag{7.9}
\]

\[
= \omega_{\mu} V^\mu \tag{7.10}
\]

where the repeated indices here are invoking the Einstein summation notation, so we get a scalar out, as desired. On a curved manifold, it is once again necessary to consider what these basis vectors for the cotangent space actually are. It turns out [4] that these basis vectors are gradient functions:

\[
\hat{\theta}^\mu = dx^\mu \tag{7.11}
\]

The notation here is a bit odd to look at, and for a more complete explanation of why the basis vectors for the cotangent space end up being written this way, I would refer you once again to one of the many thorough sources [4, 5]. Covariant vectors transform between coordinate systems in a manner reminiscent of the transformation law for vectors. For a covariant basis vector, \( dx^\mu \), transforming into the coordinates \( x'^\mu \) we have

\[
dx'^\mu = \frac{\partial x'^\mu}{\partial x^\mu} dx^\mu \tag{7.12}
\]
where With contravariant vectors and covariant vectors at our disposal, address the concept of a tensor.

Tensors are objects maps from sets of vectors and or covectors to scalars. The covariant and contravariant components of a given tensor (with \( n \) covariant components and \( m \) contravariant components) can be specified in a particular coordinate system by the action of that tensor on basis vectors and basis covectors in that coordinate system, like so:

\[
T_{\mu_1,\mu_2,\ldots,\mu_n,\nu_1,\nu_2,\ldots,\nu_m} = T(\partial\mu_1, \partial\mu_2, \ldots, \partial\mu_n, \partial\nu_1, \partial\nu_2, \ldots, \partial\nu_m)
\] (7.13)

The transformation law for tensors follows predictably from the transformation laws for vectors and covectors states above:

\[
T'_{\mu_1',\mu_2',\ldots,\mu_n',\nu_1',\nu_2',\ldots,\nu_m'} = \frac{\partial x^{\mu_1}'}{\partial x^{\mu_1}} \frac{\partial x^{\mu_2}'}{\partial x^{\mu_2}} \ldots \frac{\partial x^{\mu_n}'}{\partial x^{\mu_n}} \frac{\partial x^{\nu_1}'}{\partial x^{\nu_1}} \frac{\partial x^{\nu_2}'}{\partial x^{\nu_2}} \ldots \frac{\partial x^{\nu_m}'}{\partial x^{\nu_m}} T_{\mu_1,\mu_2,\ldots,\mu_n,\nu_1,\nu_2,\ldots,\nu_m}.
\] (7.14)

Each index transforms independently according to its contravariance or covariance.

Earlier in this thesis, we mentioned “contracting” a tensor. This is a way in which two indicies (one covariant, one contravariant) of a tensor can be summed over, resulting in a lower rank tensor. For instance, a rank (1,3) tensor \( T \) could be contracted into a rank (0,2) tensor \( X \) like so:

\[
T'_{\mu\alpha\beta} = \sum_{i=0}^{n} T'_{i\alpha\beta} = X_{\alpha\beta}.
\] (7.15)

This is precisely what we did to arrive at the Ricci Tensor from the Reimann Curvature Tensor, and although those are both confusingly named \( R \), you can tell the difference by the number of indicies.
Bibliography


