A BRIEF INTRODUCTION TO JULIA SETS OF RATIONAL MAPS

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It is difficult to view pictures of Julia sets and Mandelbrot sets without becoming something of a Platonist. These sets seem to have a reality of their own even if they don’t reflect some part of what we glibly refer to as the real world.

- John Franks, from a review of Chaos: Making a New Science

Figure 1. Julia set of \( z^2 - (0.8 + .15i) \) from [4].

1. Introduction

In the past thirty years computer images of Julia sets and the Mandelbrot set have become very popular examples of modern mathematics. The beauty and obvious complexity of the images contrasts with the very simple polynomials whose iterative process generates these images. Julia sets hold an allure similar to that of Fermat’s Last Theorem, something which every high school algebrast has encountered yet whose proof is far beyond the reach of most students of mathematics. Just like Franks, when I first encountered images of Julia sets, I believed that I was staring at some aspect of mathematical truth. It seemed clear to me that the intricacy of the Julia set held the secret to a deeper meaning. I saw so much imaginative potential in these sets that in my sophomore creative writing class I composed a story about a demented mathematician who believed Lake of the Woods, Ontario was contained in an attracting basin of some rational map. The narrator of
my story made the assumption that I, as well as many “Chaos Theorists,” are prone to make; we assumed that the Julia sets revealed something about the natural world.

Despite their visual complexity, basic properties of these sets are quite accessible. In the introductory computer science course at Oberlin College one learns a fairly simple program that produces images of the Mandelbrot set. And in an early problem set in Complex Analysis, given a fairly rudimentary definition of the filled Julia set for the complex quadratic function, we proved that the attracting basin of $\infty$ is open and thus the filled Julia set is closed. While these endeavors demystified these fractal sets, they did not satiate my thirst for a deeper understanding, which is why, after a semester and a half of independent study in complex dynamics, I am writing this paper. But the more involved I became in this project, the less important the intricate pictures of the Julia sets became. Instead I found myself studying the illusive topology of families of complex functions and the beauty of complex analysis.

Yet this is not a personal narrative through a Julia set (although that could be an interesting paper, at least for its author) so perhaps I should explain the mathematics I will be writing about. This paper is concerned with the iteration of rational maps over the extended complex plane, $\mathbb{C}_\infty$ (also known as the Riemann sphere, $\mathbb{C}_\infty$ is the union of the complex plane and a point at infinity). That is, we are concerned with functions $R(z) = P(z)/Q(z)$ where $P(z)$ and $Q(z)$ are both polynomials, and the repeated composition of these functions with themselves. We denote the $n$th iterate of a function $R(z)$ by $R^n(z)$. In other words, $R^n(z)$ is the $n$-fold composition of $R$ with itself.

A very brief historical note will be useful to explain the focus of iteration study in this paper. As Daniel Alexander explains in his book *A History of Complex Dynamics* [2], the study of iteration begins with Newton’s method (of which a crude form was known by the Babylonians), but it was not until the 1870’s that Schröder and Cayley independently studied Newton’s method over the complex plane. While Cayley focused solely on Newton’s method, Schröder viewed this process as iteration of a complex function with the goal of searching for fixed points. Over the following decades other mathematicians continued to build on the work of Schröder and Cayley, but their work failed to create a general and global theory of iteration, a theory that could describe the behavior of all rational functions on the entire complex plane. In 1915 the French Academy of Sciences announced that it would award the 1918 *Grand Prix des Sciences Mathématiques* for
such a general and global theory. The two mathematicians who succeeded in creating the modern theory of complex dynamics were Gaston Julia, who won the contest, and Pierre Fatou who, despite withdrawing his work from the contest, received a second place prize. Both saw that iteration of any function partitions the extended complex plane into a region where the iterates are equicontinuous and a region where they are not. Today for any non-constant rational function $R$, the maximal open subset of $\mathbb{C}_\infty$ on which the family of iterates of $R$, $\{R^n : n \geq 1\}$, is equicontinuous is called the Fatou set of $R$, denoted by $F$ or $F(R)$ to emphasize the function $R$, while the complement of $F$ in $\mathbb{C}_\infty$ is called the Julia set of $R$, denoted $J$ or $J(R)$. This dichotomy was the basis of the studies of Fatou and Julia and it is the basis of this paper, one goal of which is to explain the fundamental result that the Julia set is the closure of the set of repelling periodic points. More importantly the above definition of the Fatou and Julia sets phrases the theory of iteration in terms of the theory of families of complex functions and therein lies their magic. While totally disconnected Cantor sets and nowhere differentiable arcs exist in examples intentionally constructed to have these properties, Julia sets arise more organically. As opposed to being a singular mathematical object, Julia sets are a gorgeous demonstration of the variability of complex analysis.

2. Preliminaries

To define the Fatou and Julia sets on the extended complex plane first we need to make explicit the metric we will be using. As $\mathbb{C}_\infty$ is homeomorphic to the unit sphere in $\mathbb{R}^3$ via stereographic projection, we shall work on the sphere and the metric $\sigma$ we will use is given by the length of the chord between any two points on the sphere. This is called the spherical metric (or the chordal metric), and for any points $z, w \in \mathbb{C}$ we have that

$$\sigma(z, w) = \frac{2|z - w|}{\sqrt{1 + |z|^2} \sqrt{1 + |w|^2}}$$

or, if $w = \infty$, then

$$\sigma(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}.$$

The topology this metric generates makes $\mathbb{C}_\infty$ a compact space, a fact that we will use later.

The Fatou set was defined above in terms of equicontinuity:
Definition. Let \( \mathcal{F} \) be a family of functions from the metric space \((X, d)\) to the metric space \((X', d')\). We say that the family is equicontinuous if for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that whenever \( z, w \in X \) satisfy \( d(z, w) < \delta \) then \( d'(f(z), f(w)) < \varepsilon \) for all \( f \in \mathcal{F} \).

In the context of the iteration of a rational map \( R \), the family \( \mathcal{F} \) we will be considering is always the family of iterates \( \{R^n : n \geq 1\} \) and the metric that we will be using will always be \( \sigma \). Using the above definitions, we can calculate the Fatou and Julia sets for the polynomial \( P(z) = z^2 \).

Theorem 1. The Julia set for the map \( P: z \mapsto z^2 \) is the unit circle, \( S^1 \), and the Fatou set is \( \mathbb{C}_\infty \setminus S^1 \).

Proof. First we will show that the family \( \mathcal{F} = \{P^n(z) = z^{2^n} : n \geq 1\} \) is equicontinuous at every point in the open unit disk \( D \). For any two points \( z, w \in \mathbb{C} \)

\[
\sigma(z, w) = \frac{2|z - w|}{\sqrt{1 + |z|^2} \sqrt{1 + |w|^2}} \leq 2|z - w|
\]

so if we show that \( \mathcal{F} \) is equicontinuous with respect to the Euclidean metric on \( D \) then \( \mathcal{F} \) will also be equicontinuous with respect to the spherical metric. Let \( z_0 \in D \) and let \( \varepsilon > 0 \) be given. Let \( B \) be the closed ball centered at \( z_0 \) with radius \( r < 1 - |z_0| \). So \( B \) is contained in \( D \). It is clear that the sequence \( P^n(z) \) uniformly converges to 0 on the compact set \( B \), so there is a positive integer \( N \) such that for all \( n > N \) and \( z \in B \) we have that \( |P^n(z)| = |z^{2^n}| < \varepsilon/4 \). So if \( z \in B \) then

\[
\sigma(P^n(z), P^n(z_0)) < 2|P^n(z) - P^n(z_0)| \leq 2(|P^n(z)| + |P^n(z_0)|) < \varepsilon.
\]

But the functions \( P(z), \ldots, P^{N-1}(z) \) are all continuous, so for each \( P^k(z) \) there is a \( \delta_k \in (0, r) \) such that if \( \sigma(z, z_0) < 2|z - z_0| < \delta_k \) then \( \sigma(P^k(z), P^k(z_0)) < \varepsilon \). Letting \( \delta = \min\{\delta_1, \ldots, \delta_{N-1}\} \) then if \( \sigma(z, z_0) < 2|z - z_0| < \delta \) then \( \sigma(P^n(z), P^n(z_0)) < \varepsilon \) for all positive integers \( n \) showing that \( \mathcal{F} \) is equicontinuous in the unit disk.

Using the Möbius map \( M = 1/z \), which is uniformly continuous outside of the open unit disk, we invert the Riemann sphere and use the above argument to show \( \mathcal{F} \) is equicontinuous on the domain of \( \mathbb{C}_\infty \) consisting of \( |z| > 1 \).

But for any point \( w_0 \) on the unit circle \( |P^n(w_0)| = 1 \) for all positive integers \( n \). And since every neighborhood of \( w_0 \) contains points that head toward 0 under iteration by \( P \), the family \( \mathcal{F} \) is not equicontinuous at \( w_0 \). So we have just shown that the \( F(P) = \mathbb{C}_\infty \setminus S^1 \) and \( J(P) = S^1 \). \( \square \)
While calculating this particular Julia set is not terribly difficult, it turns out that the above example is quite exceptional. Julia sets are normally not smooth curves and computing them directly from the definition is usually impossible. It was not until the late 1970's that computers became powerful enough to produce accurate images of Julia sets. On the other hand, the above example is useful to keep in mind since the dynamics of the map $z \mapsto z^2$ are familiar and intuitive. In addition to analyzing the above example, directly from the definitions we can prove the following:

**Theorem 2.** For any non-constant rational map $R$ and any $i \in \mathbb{Z}_+$ we have that $F(R^i) = F(R)$ and $J(R^i) = J(R)$.

*Proof.* Let $\mathcal{F} = \{R^n : n \geq 1\}$ be the family of iterates of $R$ and let $\mathcal{F}' = \{(R^i)^n : n \geq 1\}$ be the family of iterates of $R^i$. Since $\mathcal{F}' \subset \mathcal{F}$, wherever $\mathcal{F}$ is equicontinous, $\mathcal{F}'$ is also equicontinous, giving us $F(R) \subset F(R^i)$. To show the opposite inclusion notice that for each positive integer $j$ the function $R^j$ is a continuous function. In addition, the Riemann sphere is a compact metric space, so each $R^j$ is uniformly continuous on $\mathbb{C}_\infty$. Consider the family of functions

$$\mathcal{F}_j = \{R^j(R^i)^n : n \geq 0\}.$$

Each family $\mathcal{F}_j$ consists of the iterates of $R^i$ composed with $R^j$. Since $R^j$ is uniformly continuous on $\mathbb{C}_\infty$, given $\epsilon > 0$ there is a $\delta > 0$ such that for any $w_0 \in \mathbb{C}_\infty$ if $\sigma(w, w_0) < \delta$ then $\sigma(R^j(w), R^j(w_0)) < \epsilon$. And if $\mathcal{F}'$ is equicontinous at a point $z_0 \in \mathbb{C}_\infty$ then there is a $\rho > 0$ such that if $\sigma(z, z_0) < \rho$ then $\sigma((R^i)^n(z), (R^i)^n(z_0)) < \delta$ for all positive integers $n$. This shows that the equicontinuity of the family $\mathcal{F}'$ at $z_0$ implies that the family $\mathcal{F}_j$ is also equicontinous at $z_0$. Therefore the family

$$\bigcup_{j=0}^{i-1} \mathcal{F}_j$$

is also equicontinous at $z_0$ since each family in the finite union is equicontinous at that point. But $\mathcal{F} = \bigcup_{j=0}^{i-1} \mathcal{F}_j$ so we have just shown that $\mathcal{F}$ is equicontinous at $z_0$. This shows that $F(R^j) \subset F(R)$ and thus $F(R) = F(R^j)$. Also note that $J(R) = J(R^i)$ since the Julia set is just the complement of the Fatou set on the Riemann sphere. \(\square\)

The Fatou and Julia sets are independent of which iterates of the function one considers (which begins to hint at the relationship periodic points will have with the sets). But to see the way in which the Fatou and Julia sets constitute a fundamental division of $\mathbb{C}_\infty$ we introduce the notion of
invariance. Let $X$ be any set and consider the mapping $f : X \to X$. If $A \subseteq X$ then we say that $A$ is forward invariant if $f(A) = A$ and $A$ is backward invariant if $f^{-1}(A) = A$. If $A$ is both forward and backward invariant then we say that $A$ is completely invariant. For rational maps in general we state, but do not prove, the following result:

**Theorem 3.** For any rational map $R$ of degree greater than two if $A$ is a finite set that is completely invariant under $R$ then $A$ has at most two elements.

To prove the above theorem one notices that if $A$ is a finite completely invariant set, then $R$ acts as a permutation on the elements of $A$, and thus some iterate of $R$ fixes $A$. Then considering the inverse images of the points fixed by this iterate, and finding an upper bound on the number of critical points of a rational map via the Riemann-Hurwitz relation, one can prove the theorem. This result emphasizes the particularity of completely invariant sets for rational maps and will prove to be useful later in this paper.

The above theorem motivates the following terminology for finite completely invariant sets. A point $z_0 \in \mathbb{C}_\infty$ is said to be exceptional if the set $[z] = \{R^n(z) : n \in \mathbb{Z}\}$ is finite (the set $[z]$ is known as the orbit of $z$). We denote the set of exceptional points of a map $R$ by $E(R)$. In light of the above theorem we immediately can see that any rational map of degree at least two can have at most two exceptional points. While exceptional points will play a part in this paper, of more importance now is formalizing the dichotomy between the Fatou and Julia sets.

**Theorem 4.** Let $R$ be any rational map. Then the Fatou and Julia sets of $R$ are completely invariant.

**Proof.** It is clear that if $F$ is completely invariant then $J$ is also completely invariant. And since $R$ is surjective, given any $A \subseteq \mathbb{C}_\infty$ on which $R$ is backward invariant, $R(R^{-1}(A)) = A$. So we need only show that $F$ is backward invariant.

First we will show that $R^{-1}(F) \subseteq F$. Let $z_0 \in R^{-1}(F)$ and let $w_0 = R(z_0)$. As $w_0 \in F$, the family $\{R^n\}$ is equicontinuous at $w_0$. Hence for every $\epsilon > 0$ there is a $\delta > 0$ such that if $\sigma(w, w_0) < \delta$ then $\sigma(R^n(w), R^n(w_0)) < \epsilon$ for all natural numbers $n$. And since $R$ is continuous there is a $\rho > 0$ such that if $\sigma(z, z_0) < \rho$ then $\sigma(R(z), R(z_0)) = \sigma(R(z), w_0) < \delta$, which implies that the family $\{R^{n+1} : n \geq 1\}$ is equicontinuous at $z_0$. As in the proof of Theorem 2 we have that the entire family of iterates of $R$ is equicontinuous at $z_0$. Therefore $R^{-1}(F) \subseteq F$. 

The proof of the opposite inclusion is similar. Let $z_0 \in F$ and let $w_0 = R(z_0)$. Since the family \{\{R^n\}\} is equicontinuous at $z_0$, given any $\epsilon > 0$ there is a $\delta > 0$ such that for all $n \in \mathbb{Z}_+$ if $\sigma(z, z_0) < \delta$ then $\sigma(R^{n+1}(z), R^{n+1}(z_0)) < \epsilon$. Let $N = \{z: \sigma(z, z_0) < \delta\}$. As $R$ is non-constant and analytic, it is an open mapping, so $R(N)$ contains an open neighborhood of $w_0$. If $w \in R(N)$ then $w = R(z)$ for some $z \in N$, meaning that $\sigma(R^n(w), R^n(w_0)) = \sigma(R^{n+1}(z), R^{n+1}(z_0)) < \epsilon$. Therefore the family \{\{R^n: n \geq 1\}\} is equicontinuous at $w_0$, implying $w_0 \in F$. This shows that all points in $F$ are mapped to other points in $F$ by $R$. This gives that $F \subset R^{-1}(F)$, completing the proof.

Before ending this section there is one last preliminary definition that should be mentioned.

**Definition.** Given an analytic map $R$, a point $z \in \mathbb{C}_\infty$ is said to be a *periodic point* if $R^n(z) = z$ for some positive integer $n$. Periodic points are classified as follows:

(a) *attracting* if $|(R^n)'(z)| < 1$;

(b) *repelling* if $|(R^n)'(z)| > 1$;

(c) *indifferent* if $|(R^n)'(z)| = 1$.

3. Normal Families

The results discussed so far rely on the straightforward geometric and topological notions of equicontinuity. However the related notion of a normal family of functions is often used to define the Fatou and Julia sets because nearly all of the more profound insights rely upon deep results in the theory of normal families.

**Definition.** Let $C(G, \Omega)$ be the set of all continuous functions from a domain\(^1\) $G$ to a complete metric space $\Omega$. A family of functions $\mathcal{F} \subset C(G, \Omega)$ is *normal* if each sequence in $\mathcal{F}$ has a subsequence which converges uniformly on every compact subset of $G$.

In fact, as Alexander argues, it was Paul Montel’s creation of a theory of normal families that provided results powerful enough to deal with the iteration of functions of a complex variable. Although Montel himself developed the theory to study the convergence of sequences and series of analytic functions and to study the Picard theorems, he regarded Fatou and Julia’s papers as “one of the most important applications of the theory of normal families” \[2\]. The connection

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\(^1\)By domain I mean a non-empty open, connected subset of $\mathbb{C}_\infty$. 

between the Fatou and Julia sets and the theory of normal families lies in the connection between equicontinuity and normality. In fact, in many situations, the two properties are equivalent, as stated in the following theorem:

**Arzela-Ascoli Theorem.** Let $D$ be a domain in $\mathbb{C}_\infty$. A family $F$ of continuous functions with values in a metric space $\Omega$ is normal on $D$ if and only if:

1. For any $z$ in $D$ the values $f(z)$, for all $f \in F$, lie in a compact subset of $\Omega$; and
2. $F$ is equicontinuous on every compact set $E \subset D$.

Since in our situation $\Omega$ is the Riemann sphere with the spherical metric, condition (a) is automatically fulfilled. The proof of this theorem is quite involved, making use of a metric on the space of all continuous functions, and its inclusion in this paper would be a significant digression. There is a full proof on page 214 of [1] which contains an elegant diagonal argument. But this connection between equicontinuity and normality allows us to use a number of results from the theory of normal families, perhaps none more important to iteration theory than Montel’s theorem.

**Montel’s Theorem.** Let $F$ be a family of meromorphic functions on a domain $D \subset \mathbb{C}_\infty$. If there are three values $\alpha, \beta, \gamma \in \mathbb{C}_\infty$ such that no $f \in F$ takes any of the values $\alpha, \beta$, or $\gamma$, then $F$ is a normal family.

This theorem is essential in the discussion of Fatou and Julia sets since it presents a very simple criterion for normality. The proof of this theorem, however, relies upon Marty’s theorem—another normality criterion that asserts that a family is normal if and only if a quantity called the spherical derivative is locally bounded—and a result known as Zalcman’s principle (or one can prove Montel’s theorem using certain modular functions).\(^2\) So a proof of this theorem will not be contained in this paper. It is also not too difficult to show that the values $\alpha, \beta$, and $\gamma$ in the hypothesis of Montel’s theorem can vary with the functions in the family (so long as these three values always are distinct and pairwise a positive distance apart). One just uses a family of Möbius maps to map each function’s three omitted values to some $\alpha, \beta$, and $\gamma$, and we can apply Montel’s theorem to this new family. Since the Möbius maps are uniformly continuous this transfers the equicontinuity of the new family to the original family, giving us normality. Throughout the paper I will be referencing this corollary simply as Montel’s theorem.

\(^2\)A proof of this theorem and related lemmas is contained on page 76 of [8].
4. Topology of the Julia Set

Before we use Montel’s theorem to unravel the mathematical properties of the Julia set, first let us show that for rational maps of degree at least two (an assumption that we will be making from here on out), the Julia set contains at least three points. If for a rational map \( R \) we have that the Julia set \( J \) is empty, then the family \( \{ R^n : n \geq 1 \} \) is normal on all of \( \mathbb{C}_\infty \). So on every compact set \( K \subset \mathbb{C}_\infty \) there is a convergent subsequence, \( \{ R^{n_k} : n_k \in \mathbb{Z}_+ \} \). A theorem of Adolf Hurwitz’s states that if a sequence of functions \( f_n \) converges to \( f \) on a domain \( G \) then, under certain constraints, there is a positive integer \( N \) such that for all \( n > N \) the functions \( f_n \) and \( f \) have the same number of zeroes.\(^3\) This implies that for all \( n_k \geq N \) the function \( R^{n_k} \) and the limit function have the same degree. However the degree of \( R^{n_k} \), \( \deg(R^{n_k}) \), is the same as the degree of \( R \), \( \deg(R) \), to the \( n_k \)th power. But if \( \deg(R^{n_k}) = (\deg(R))^{n_k} \) for an infinite number of \( n_k \), then we must have that \( \deg(R) \leq 1 \), contrary to our assumption.

Now we know that \( J \) contains at least one point. Because \( J \) is completely invariant, if \( J \) is finite it can only contain exceptional points of which there are at most two. Assume that there is only one point \( z_0 \in J \). We can conjugate \( R \) by a Möbius map \( g \) and assume that \( z_0 = \infty \) and in addition we note that this conjugation preserves the Julia and Fatou sets\(^4\). Since \( \infty \) is an exceptional point, we must have that the map is a polynomial, for if it had any poles then \( \infty \) would no longer be an exceptional point. But \( \infty \) is clearly in the Fatou set of any polynomial, so \( J \) cannot have only one point. So assume that \( J \) contains only two points, \( z_0 \) and \( z_1 \). Similarly, we can conjugate by a Möbius map \( g \) and assume that \( z_0 = 0 \) and \( z_1 = \infty \). This means that the conjugation of \( R \) behaves like some polynomial \( az^n \) where \( a \in \mathbb{C} \) and \( n \in \mathbb{Z} \). But for this map \( 0 \) and \( \infty \) are in the Fatou set contradicting our assumption that they are in \( J \), so \( J \) cannot have only two points. We have just shown that the exceptional points of a rational map lie in the Fatou set. But more importantly we have shown that \( J \) is infinite.

Now we are ready to prove our first characterization of the Julia set which we will do in terms of completely invariant sets.

**Theorem 5.** Given a rational map \( R \) of degree at least two let \( A \) be a closed subset of \( \mathbb{C}_\infty \) that is completely invariant under \( R \). Then either \( A \subset E(R) \subset F \) or \( J \subset A \).

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\(^3\)The full statement and proof of this theorem can be found on page 152 of [5].

\(^4\)That is \( J = g(R(g^{-1}(J))) \). This is proved on page 50 of [3].
Proof. Consider the set $U = \mathbb{C}_\infty \setminus A$. We know that if $A$ is finite then it contains no more than two elements. These elements are necessarily exceptional points and by the above argument they lie in the Fatou set. So let us assume that $A$ is infinite. Since $A$ is completely invariant under $R$, so is $U$. Consider the family of functions $\mathcal{F} = \{R^n : n \geq 1\}$ on the open set $U$. Since $U$ is completely invariant, for any $z \in U$ and any positive integer $n$ we have that $R^n(z) \notin A$. Since $A$ contains more than three elements, we know that each iterate of $R$ omits at least three values over $U$. Montel's theorem allows us to conclude that $\mathcal{F}$ is normal on $U$. So $U \subset F$ implying that $J \subset A$. □

The proof of the above theorem was quite painless and this will be a theme throughout the remainder of this paper. The real work behind most of these results is contained in the machinery of Montel's theorem. Also notice that this theorem allows us to see the Julia set as the minimal closed completely invariant set with at least three points. This theorem highlights the duality between the Julia set and the Fatou set, which is the maximal open set on which a map is equicontinuous or normal. There are a couple of other topological properties of the Julia set that are relatively simple to demonstrate. We collect them in the following theorem.

**Theorem 6.** Let $R$ be a rational map of degree at least two. $J$ is a perfect set and either $J = \mathbb{C}_\infty$ or $J$ has empty interior.\(^5\)

Proof. First we will show that $J$ is perfect. Let $J'$ be the set of all limit points of $J$. Since $J$ is closed $J' \subset J$. Also, because $J$ is an infinite closed subset of a compact space, $J$ is compact and therefore $J'$ is non-empty. We shall now show that $J'$ is completely invariant, and since $R$ is surjective, we need only show that $J'$ is backward invariant. For any $z \in J'$ there is a sequence of points $\{z_n\}$ in $J$ such that as $n \to \infty$ the points $z_n \to z$. Because $R$ is continuous, the sequence $\{R(z_n)\}$ converges to $R(z)$. Therefore $R(z) \in J'$ and $R(J') \subset J'$, implying $J' \subset R^{-1}(J')$.

To show the opposite inclusion, we consider a point $z \in R^{-1}(J')$ and let $w = R(z)$. Since $R$ is an open map, given an open set $U$ about $z$, its image $R(U)$ is an open neighborhood containing $w$. Because $w \in J'$ the intersection $R(U) \cap J$ is non-empty. But then

$$R^{-1}(R(U) \cap J) = (R^{-1}(R(U))) \cap (R^{-1}(J)) = U \cap J$$

\(^5\)Often computer images of the Julia sets of polynomials are not depicted with empty interior. This is because these images are of the filled Julia sets which are $\mathbb{C}_\infty \setminus F_\infty$ where $F_\infty$ denotes the component of the Fatou set containing $\infty$. Since for any polynomial $\infty$ is an attracting fixed point, all points in $F_\infty$ limit towards it. To generate an approximate image of the filled Julia sets, one can simply iterate all points in a domain and if after a certain number of iterations a point surpasses a set bound, then it is in $F_\infty$ and otherwise it is “in” the filled Julia set.
is also non-empty, so \( z \in J' \). This shows that \( R^{-1}(J') \subset J' \) and we conclude that \( J' \) is completely invariant. In addition \( J' \) only contains limit points of \( J \) so it is closed. Therefore we have shown that \( J' \) is a closed, completely invariant set disjoint from \( F \) and Theorem 5 implies that \( J \subset J' \). This shows that \( J = J' \) making \( J \) a perfect set.

To show the second part of the theorem notice that the Riemann sphere is the disjoint union of the interior of the Julia set \( \text{int}(J) \), the boundary of the Julia set \( \partial(J) \), and the Fatou set \( F \). It is not very difficult to show that since \( J \) is completely invariant so are \( \text{int}(J) \) and \( \partial(J) \). If \( F \neq \emptyset \), then \( F \cup \partial J \) is an infinite closed, completely invariant set. Theorem 5 asserts that \( J \subset (F \cup \partial J) \) which means that \( J \subset \partial J \) and \( \text{int}(J) = \emptyset \). □

An immediate consequence of Theorem 6 is that the Julia set is uncountably infinite. Yet despite the topological restrictions on the Julia set it can actually take on a wide variety of shapes. Earlier in the paper we showed that the Julia set of \( z^2 \) is the unit circle and in Theorem 6 we implied that there are rational maps for which \( J = \mathbb{C}_\infty \). Samuel Lattés, a French mathematician who also participated in the 1918 Grand Prix, discovered the following example of such a function,

\[
l(z) = \frac{(z^2 + 1)^2}{4z(z^2 - 1)},
\]

which is known as Lattés function. However, in most situations, the Julia set is a far more complex object. A computer graphic of the Julia set for the complex quadratic function \( z^2 - 1 \) is shown in Figure 2. Without proof we note that the image appears to be a self-similar fractal containing no smooth arcs, which is vastly different from the unit circle.

![Figure 2. Julia set of \( z^2 - 1 \) from [9].](image-url)
Yet Julia sets can be even more irregular, even for functions as simple as the complex quadratic function $Q_c = z^2 + c$. We now offer a heuristic argument that the Julia sets of $Q_c$ are Cantor sets whenever $|c| > 2$.\footnote{The full set of $c$ values for which the Julia set is a Cantor set is the complement of the Mandelbrot set.} Let $A$ be the set of points in $\mathbb{C}_\infty$ whose forward orbits lie entirely within the disk $\{z : |z| \leq |c|\}$. We will show that $A$ is a completely invariant Cantor set, so Theorem 5 implies that $J \subset A$. And we will also show that every other point in $\mathbb{C}_\infty$ is attracted to $\infty$. And since $A$ is a Cantor set contained in the disk of radius $|c|$, the iterates of $Q_c$ cannot be equicontinuous at any point of $A$ so $A \subset J$ and thus $A = J$.

First we argue that for all $|z| \geq |c| > 2$ the forward images $Q_c^n(z) \to \infty$ as $n \to \infty$. Notice that

$$|Q_c(z)| \geq |z|^2 - |c| \geq |z|^2 - |z| = |z(|z| - 1)|.$$  

Since $|z| - 1 > 1$ we have that $|Q_c(z)| > k|z|$ where $k > 1$ and the claim follows by induction.

Let $\gamma$ be the circle $|z| = |c|$. Under $Q_c$ all points on $\gamma$ have two pre-images except for $c$ whose only pre-image is 0. So $Q_c^{-1}(\gamma)$ is a figure-eight curve that is contained in the interior of the disk $\{z : |z| \leq |c|\}$. Also notice that all points between $Q_c^{-1}(\gamma)$ and $\gamma$ are mapped outside of $\gamma$ and the above claim implies that they are attracted to $\infty$. This is pictured below.

![Figure 3. Arc of $\gamma$ and its preimages under $Q_c^{-1}$ and $Q_c^{-2}$ from [6].](image)

Let $D$ be any disk contained inside $\gamma$ that also contains $Q_c^{-1}(\gamma)$. First, note that $A = \bigcap_{n=1}^{\infty} Q_c^{-n}(D)$. And the pre-image of $D$ consists of two simply connected figure-eight sets, one in each lobe of $Q_c^{-1}(\gamma)$. Inductively, the sequence of sets $Q_c^{-n}(D)$ consists of $2^n$ nested compact and connected sets. And the infinite intersection of these nested compact, connected sets is a unique point (a fact which is not too difficult to show). So without belaboring the point any longer we see that $A$ has

\footnote{This argument is based off of a theorem on page 270 of [6].}
the structure of a Cantor set. We have seen Julia sets for the complex quadratic function vary from smooth curves, to non-differentiable closed curves, to a totally disconnected Cantor set. This result generalizes to the theorem:

**Theorem 7.** If $J$ is disconnected then it has uncountably many components and each point of $J$ is a limit point of infinitely many distinct components of $J$.

While we will not prove this theorem (it relies upon an involved lemma about the topological properties of rational maps) it shows that the above example is not unique. Whenever the Julia set is disconnected it will be a Cantor set. Beardon shows that for the map $R(z) = z^2 + \lambda/z^3$, whenever $\lambda$ is sufficiently small the Julia set is a Cantor set of circles[3]. So although not every disconnected Julia set is a totally disconnected perfect set, every disconnected Julia set is still an intricate perfect set.

In addition, we note that there is a similar theorem about the number of components of the Fatou set.

**Theorem 8.** The Fatou set of a rational map has either infinitely many components or at most two components.

In this section we have seen that the Fatou set of Lattés function has zero components, the Fatou set of $Q_c$ has one component (when $|c| > 2$), The Fatou set of $z^2$ has two components, and the Fatou set of $z^2 - 1$ has infinitely many components. Even though this paper is focused on descriptions of the Julia set, there is a nice duality between the Fatou and Julia sets, which the above theorem and examples highlight. In the only example for which the Fatou set is connected, $Q_c$ for $|c| > 2$, the Julia set is totally disconnected. And in the only example for which the Fatou set has infinitely many components, $z^2 - 1$, the Julia set is a connected curve.

5. **Further Properties of the Julia Set**

Regardless of its topology, there are still quite a few properties that are fundamental to the Julia Set. Before we show that the Julia set is the closure of the set of repelling periodic points we will prove another theorem that highlights the repelling nature of the Julia set. Consider the Julia set for $P(z) = z^2$, which we know is the unit circle. Clearly any point not in the Julia set either converges to 0 or $\infty$. But consider an open neighborhood of a point on the Julia set, for example
the open disc \( D = \{ z : |z - 1| < 1/2 \} \). The image \( P(D) \) is a larger open set that strictly contains \( D \). And if we repeat this process indefinitely it will give rise to larger and larger open sets that will eventually contain every point in \( C_\infty \) except for 0 and \( \infty \). In this way the Julia set acts like a repeller, as every neighborhood around the Julia set eventually covers most of \( C_\infty \) under iteration.

The qualification “most” is used because for a rational map \( R \) these open sets will miss the set of exceptional points \( E(R) \). This result is generalized and proved below.

**Theorem 9.** Let \( R \) be a rational map of degree at least two and let \( U \) be any non-empty open set such that \( U \cap J \neq \emptyset \). Then:

(a) \( \bigcup_{n=0}^{\infty} R^n(U) \supset C_\infty \setminus E(R) \);

(b) \( R^n(U) \supset J \) for all sufficiently large integers \( n \).

**Proof.** First we prove condition (a). Let \( U_0 = \bigcup_{n=0}^{\infty} R^n(U) \) and let \( V = C_\infty \setminus U_0 \). Assume that \( V \) has at least three points. Then Montel’s Theorem tells us that the family of iterates \( \{ R^n : n \geq 1 \} \) is normal on \( U_0 \). But \( J \cap U_0 \neq \emptyset \) so the family \( \{ R^n : n \geq 1 \} \) cannot be normal on \( U_0 \). Thus \( V \) has at most two points. Let \( z_0 \in V \) and assume that \( z_0 \) is not an exceptional point. We must have that the set \( O^-(z_0) = \{ R^{-n}(z_0) : n \geq 1 \} \) is infinite. By the above argument there is some \( w_0 \) such that \( w_0 \in O^-(z_0) \cap U_0 \). So there is some positive integer \( k \) such that \( R^k(w_0) = z_0 \) and another positive integer \( l \) such that \( w_0 \in R^l(U_0) \). We conclude that \( z_0 \in R^{k+l}(U_0) \) contradicting our assumption that \( z_0 \in V \). So all points in \( C_\infty \), with the possible exception of \( E(R) \), are contained in \( U_0 \), thereby proving (a).

To show (b) let \( U_1, U_2, \) and \( U_3 \) each be open sets contained in \( U \) whose intersection with \( J \) is non-empty. In addition, assume that the sets are separated and that there is a positive distance between each set. Let us also define the set \( A = \{ 1, 2, 3 \} \). First we argue that for each \( i \in A \) there is a \( j \in A \) and a positive integer \( n \) such that \( U_j \subset R^n(U_i) \). In other words, we show that some image of \( U_i \) covers \( U_j \). If this were not the case for some \( U_i \), then we could find points \( u_1 \in U_1, u_2 \in U_2, \) and \( u_3 \in U_3 \) such that for all positive integers \( n \) and all \( k \in A \) we have that \( u_k \notin R^n(U_i) \). Montel’s Theorem implies that \( \{ R^n : n \geq 1 \} \) is normal on \( U_i \), a contradiction since \( U_i \cap J \neq \emptyset \).

Let us introduce a map \( \pi : A \to A \) where \( \pi(i) \) is the integer \( j \in A \) such that \( U_j \subset R^n(U_i) \). This maps \( A \) into \( A \) so some iterate of \( \pi \) must have a fixed point. That is, there is some \( i \in A \) and an integer \( n_0 \) such that \( U_i \subset R^{n_0}(U_i) \).
Let $S = R^{m_0}$. The sequence $U_i \subset S(U_i) \subset S^2(U_i) \subset \cdots$ is an increasing sequence of nested open sets. Since $U_i \cap J \neq \emptyset$, part (a) of the theorem tells us that the set $\bigcup_{m=0}^{\infty} S^m(U_i)$ forms an open cover of $J$. Since $J$ is compact and the $S^m$ are nested, there is an integer $m_0$ such that $J \subset S^{m_0}(U_i)$. And since $U_i \subset U$ we have that $J \subset S^{m_0}(U_i) \subset S^{m_0}(U)$. But notice that $S^{m_0} = R^{m_0}$. Letting $N = n_0 m_0$ we have that $J \subset R^N(U)$. So for all positive integers $s$ we have that $R^s(J) \subset R^s(R^N(U)) = R^{N+s}(U)$. Since the Julia set is completely invariant $R^s(J) = J$ and for all integers $n \geq N$, 

$$J \subset R^n(U)$$

completing the proof of (b).

While the above theorem shows that open sets containing the Julia eventually cover most of $\mathbb{C}_\infty$, a sort of converse is true as well. Once again, think of the quadratic map $P(z) = z^2$. Given any point $z \neq 0, \infty$, if one repeatedly takes square roots of $z$, then eventually these points limit on the unit circle. This is expressed more formally in the following theorem concerning the backward orbits of non-exceptional points, where the backward orbit is the set $O^- = \{ R^{-n}(z) : n \geq 1 \}$.

**Theorem 10.** For a rational map $R$ of degree greater than two if $z$ is not an exceptional point, then $J$ is contained in the closure of $O^-(z)$. When $z \in J$ then the Julia set is the closure of $O^-(z)$.

**Proof.** Given an open set $U$ such that $U \cap J \neq \emptyset$ we want to show that $U$ contains some point in $O^-(z)$. Theorem 9 tells us that since $z$ is not exceptional there is a positive integer $n$ such that $z \in R^n(U)$. Since $R^{-n}(z) \in O^-(z)$ we have just shown that $U$ contains a point in $O^-(z)$, proving that $J$ is contained in the closure of $O^-(z)$. And if $z \in J$, then because $J$ is completely invariant, $O^-(z) \in J$. In addition $J$ is a closed set so $J$ contains the closure of $O^-(z)$ as well. And by the preceding argument we get that the $J$ is the closure of $O^-(z)$. 

This theorem tells us that the backward orbit of nearly any point in $\mathbb{C}_\infty$ traces out the Julia set in the limit. More importantly, Theorem 10 motivates the following algorithm for computing Julia sets. First one takes a point $z$ which is not an exceptional point or a critical point and then takes its inverse image $R^{-1}(\{z\})$. Since $R$ is of degree $d$ the set $R^{-1}(\{z\})$ has $d$ elements. Thus if we repeat this process, after a finite number of iterations $N$ all of the points in $R^{-n}(\{z\})$ will be very close to $J$ for $n > N$. An issue with this algorithm is that the size of $R^{-n}(\{z\})$ blows up very quickly, but this can be solved in some situations (as when $R$ is a complex quadratic) by randomly
choosing one point in each pre-image to use in the algorithm. While this result was known to Julia and Fatou, in the days before computers calculating the sets $R^{-n}(\{z\})$ was very difficult.

Now we are ready to consider the role that periodic points have in the Julia set. While the theorem below implies that there are infinitely many periodic points, with a few lemmas it will prove significantly more. First we introduce the following definition. Given a set $A$, the derived set of $A$ is the set of all limit points of $A$.

**Theorem 11.** Let $R$ be a rational map of degree $d$ where $d \geq 2$. Then the Julia set is the derived set of the periodic points of $R$.

**Proof.** Let $U$ be an open set with $J \cap U \neq \emptyset$. We will show that there is a periodic point in $U$. Let the point $z_0 \in J \cap U$ be such that $z_0$ is not the image of a critical point for $R^2$. Since $R$ is a $d$-fold map at $z_0$, and $d$ is at least two, the set $R^{-2}(\{z_0\})$ contains at least four distinct points, and there are at least three points $z_1, z_2, z_3$ that are distinct from each other and $z_0$. Around each $z_i$ we can construct a neighborhood $U_i$ subject to the following constraints:

(a) the $U_i$ have pairwise disjoint closures;

(b) $U_0 \subset U$ and $U_0 \cap J \neq \emptyset$;

(c) for each $j \in A$ the function $R^2$ is a homeomorphism of $U_j$ onto $U_0$.

Let $S_j : U_0 \rightarrow U_j$ be the inverse of $R^2 : U_j \rightarrow U_0$. If there is a periodic point $\zeta$ in $U_0$ then for some integer $n$ we have that $R^n(\zeta) = \zeta$, so for some $j \in A$ we have that $R^{n-2}(\zeta) = S_j(\zeta)$. Assume that there are no periodic points in $U_0$. Then for all $z \in U_0$, all $j \in A$ and all positive integers $n$ we have that $R^n(z) \neq S_j(z)$. But then on $U_0$ each function in the family $\{R^n : n \geq 1\}$ omits three distinct points, one in each $U_i$ for $i \in A$. This implies that the family $\{R^n : n \geq 1\}$ is normal on $U_0$, a contradiction since $U_0 \cap J \neq \emptyset$. So there is a periodic point in $U_0$ and thus there is a periodic point in every open set $U$ that meets $J$. This means that $J$ is contained in the derived set of the periodic points of $R$.

To show that the derived set of periodic points is contained in $J$ we use the fact, which we will not prove, that each component of the Fatou set contains at most one periodic point of $R$. So all periodic points which lie in $F$ have no limit points, proving the theorem.

$\square$
We are now a few steps away from the completion of the papers last theorem. First note that in every neighborhood of a repelling fixed point of $R$, the family $\{R^n : n \geq 1\}$ fails to be equicontinuous merely because points in that neighborhood are moving away from the point that is staying fixed. So repelling fixed points lie in $J$ and, since periodic points are fixed points of some $R^n$ and $J(R^n) = J(R)$, we have that repelling periodic points also lie in $J$.

Last we mention that rational maps have at most $2d - 2$ non-repelling cycles where $d$ is the degree of $R$. The proof of this theorem is beyond the scope of this paper, but this fact along with Theorem 11 tells us that the Julia set is the derived set of the repelling periodic points. And combining this with the above argument gives us that

**Theorem 12.** The Julia set is the closure of the set of repelling periodic points.

This result provides a characterization of the Julia set that gives a feel of both the topology and dynamics of the set. Yet the path that we have taken in this paper, first defining the Julia set in terms of equicontinuity and then arriving at this conclusion, is only one way to investigate the dynamics of rational maps. While the original paper of Fatou followed this approach, Julia first considered the set of repelling periodic points and showed that the closure of this set is exactly the set of points in $\mathbb{C}_\infty$ which lie in no open sets where the family $\{R^n : n \geq 1\}$ is normal. Yet for both Fatou and Julia the key was Montel’s theory of normal families. Earlier in the paper I mentioned Lattés, another pioneer in complex dynamics. While he is still credited with discovering a rational map with empty Fatou set, his work described neither the way in which $\mathbb{C}_\infty$ is partitioned into domains where there are attractors nor the boundaries between these domains. Instead of using the theory of normal families to study periodic points Lattés used functional equations. Alexander argues that because Lattés did not use normal families he must have been ignorant of them. Since if he was aware of Montel’s theory he would immediately have leapt from his results to those of Fatou and Julia. To understand complex iteration fully one must harness the full power of complex analysis. And more importantly, despite the beauty of the images, they are not the harbinger of a new type of knowledge. Rather, the images are a metaphor for the complex relationship between the computer and modern science and mathematics. While the computers allow for visualization the mathematics lies in a different, older realm.
REFERENCES


Many of the proofs included in this paper are based on proofs contained in this book and my approach is very closely based on Beardon's.


